Rate Splitting is Approximately Optimal for Fading Gaussian Interference Channels

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***Abstract*— In this paper, we study the 2-user Gaussian interference-channel with feedback and fading links. We show that for a class of fading models, when no channel state information at transmitter (CSIT) is available, the rate-splitting schemes for static interference channel, when extended to the fading case, yield an approximate capacity region characterized to within a constant gap. We also show a constant-gap capacity result for the case without feedback. Our scheme uses rate-splitting based on average interference-to-noise ratio (inr). This scheme is shown to be optimal to within a constant gap if the fading distributions have the quantity $\log\left(\mathbb{E}\left[inr\right]\right) - \mathbb{E}\left[\log\left(inr\right)\right]$ uniformly bounded over the entire operating regime. We show that this condition holds in particular for Rayleigh fading and Nakagami fading models. The capacity region for the Rayleigh fading case is obtained within a gap of 2.83 bits for the feedback case, and within 1.83 bits for the non-feedback case.**

I. Introduction

The 2-user Gaussian interference channel (IC) is a simple model that captures the effect of interference in wireless networks. Significant progress has been made in the last decade in understanding the capacity of the static Gaussian IC. But in practice the links in the channel could be varying rather than static. In this paper, we study the 2-user Gaussian IC with fading links.

Previous works have characterized the capacity region to within a constant gap for the static Gaussian IC with and without feedback. The capacity of the 2-user Gaussian IC without feedback was characterized to within 1 bit in [3]. In [7], Suh *et al.* characterized the capacity of the Gaussian IC with feedback to within 2 bits. These results were based on the Han-Kobayashi scheme, where the transmitters split their messages into common and private parts.

To the best of our knowledge no previous work has provided a simple scheme to completely characterize the capacity region for the case of the continuously fading channel with no channel state information at transmitter (CSIT). The general Han-Kobayashi scheme for discrete memoryless IC [2] indeed holds for the problem at hand, but it is very complex due to the time-sharing involved. In [10], Wang *et al.* considered the bursty IC where the interference is either present or not present. In [9], Vahid *et al.* studied the binary fading model for the two-user interference channel, where the channel gains, the transmit signals and the received signals are in the binary field. In [5], Gou *et al.* proposed an interference neutralization scheme and showed a 2 degrees of freedom result for $2\times2\times2$ fading interference channel with full channel state information at the relays and destinations. In [6], Kang *et al.* considered interference alignment for the fading K-user IC with delayed feedback and showed a result of $\frac{2K}{K+2}$ degrees of freedom. Tuninetti [8] studied power allocation policies for fading Gaussian IC with CSIT and numerically showed that for Rayleigh fading, their scheme is close to optimal for some system parameters. In [4] Farsani showed a one bit capacity result for fading Gaussian IC with partial CSIT. Their scheme [4] employs dynamic power allocation depending on the available CSIT to achieve any given rate point and requires a union of all power allocation policies for both transmitters to achieve the whole inner bound.

In this paper, we show that the Han-Kobayashi type rate-splitting schemes [2], [3], [7] can be extended to a class of fading models that satisfy a condition on the distribution of crosslink strengths. We will show that rate-splitting based on $\frac{1}{inr}$ for the static case in [3], [7] can naturally be extended to schemes with rate-splitting based on $\frac{1}{\mathbb{E}[inr]}$ for the fading case to approximately obtain the whole capacity region. Our schemes use fixed power allocation to achieve any given rate point, and to achieve the whole inner bound for the feedback case we need to vary only a single power allocation parameter (other than choosing the common and private message rates), inheriting these properties from [3], [7]. The condition on distribution of crosslink strengths is required to ensure that the splitting is indeed optimal within a constant gap. In particular, we will show that common fading models, including Rayleigh and Nakagami fading, satisfy the required condition.

The paper is organized as follows. In section II we set up the problem and explain the notations used. In section III we discuss the main results of our paper, in section IV we go through the details of the scheme for the feedback case, in section V we go through the results for the non feedback case, and in section VI we show that common fading models including Rayleigh and Nakagami fading satisfy the required conditions.

II. Notation

We consider the 2-user Gaussian fading IC

$$Y_1 = g_{11}X_1 + g_{21}X_2 + Z_1$$
$$Y_2 = g_{22}X_2 + g_{12}X_1 + Z_2$$

where the links g_{ij} are fading, the realizations of g_{ij} for any fixed (i,j) are i.i.d across time, and the realizations for different (i,j) are independent. The instantaneous interference-

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to-noise ratios (inr) are given by $|g_{21}|^2$ and $|g_{12}|^2$. We assume that the transmitters has no knowledge of the channel states before transmission. For the feedback case, after each reception, each receiver reliably feeds back the received symbol and the channel states to its corresponding transmitter, for example the receiver 1 feeds back (Y_1, g_{11}, g_{21}) to transmitter 1. We normalize the signal power and noise power to 1, i.e., $P_k = 1, Z_k \sim \mathcal{CN}(0,1)$. We denote $\mathbb{E}\left[|g_{ii}|^2\right]$ as SNR_i and $\mathbb{E}\left[|g_{ij}|^2\right]$ when $i \neq j$, as INR_i. We use the vector notation $\underline{g_1} = [g_{11}, g_{21}]$, $\underline{g_2} = [g_{22}, g_{12}]$ and $\underline{g} = [g_{11}, g_{21}, g_{22}, g_{12}]$. For a complex number z, we use $\mathrm{Re}(z)$ to indicate its real part. The natural logarithm is denoted by $\ln()$ and the logarithm with base 2 is denoted by $\log()$.

III. Summary of main results

For a given wireless system, the signal strengths would have some probability distribution; depending on the operating conditions, the distributions might vary and hence we get a class of distributions. We identify $W = |g_{ij}|^2$ for ease of notation. Suppose the class of distributions satisfy

$$\log(a + \mu_W) - \mathbb{E}[\log(a + W)] \le c \tag{1}$$

for some constant c independent of the distribution, for all $a \ge 0$, then we claim that we have the following constant gap capacity characterization for the feedback and non-feedback cases. We assume that we have the same c for both crosslinks, but our results can be easily modified to the case when there are two different c_1 and c_2 for the two crosslinks.

The following two theorems summarize our results. The achievability schemes for both cases are Han-Kobayashi schemes with rate-splitting based on $\mathbb{E}[inr]$. For both cases we obtain capacity gaps in terms of the constant c from condition (1), where a capacity gap of δ means that for any rate pair (R_1, R_2) in the outer bound region, the rate pair $(R_1 - \delta, R_2 - \delta)$ is contained in the inner bound region

Theorem 1. *For the fading interference channel with feedback, the rate region described by* (2) − (7) *is achievable for* $0 \le |\rho|^2 \le 1$, $0 \le \theta < 2\pi$ *with* $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1 - |\rho|^2\right)$*:*

$$R_1 \le \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 Re\left(e^{i\theta} g_{11} g_{21}^*\right) + 1\right)\right] - 1 \tag{2}$$

$$R_1 \le \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \tag{3}$$

$$R_2 \le \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2|\rho|^2 Re\left(g_{22}^* g_{12} e^{i\theta}\right) + 1\right)\right] - 1 \tag{4}$$

$$R_2 \le \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 2 \tag{5}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2|\rho|^2 Re\left(g_{22}^* g_{12} e^{i\theta}\right) + 1\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \tag{6}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 Re\left(e^{i\theta} g_{11} g_{21}^*\right) + 1\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 2 \tag{7}$$

An outer bound for the feedback case is given by (8) − (13) *for complex number* ρ *with* $0 \le |\rho| \le 1$*:*

$$R_1 \le \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2Re\left(\rho g_{11} g_{21}^*\right) + 1\right)\right] \tag{8}$$

$$R_1 \le \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{11}|^2}{1 + \left(1 - |\rho|^2\right)|g_{12}|^2}\right)\right] \tag{9}$$

$$R_2 \le \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2Re\left(\rho g_{22}^* g_{12}\right) + 1\right)\right] \tag{10}$$

$$R_2 \le \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{22}|^2}{1 + \left(1 - |\rho|^2\right)|g_{21}|^2}\right)\right] \tag{11}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2Re\left(\rho g_{22}^* g_{12}\right) + 1\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{11}|^2}{1 + \left(1 - |\rho|^2\right)|g_{12}|^2}\right)\right] \tag{12}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2Re\left(\rho g_{11} g_{21}^*\right) + 1\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{22}|^2}{1 + \left(1 - |\rho|^2\right)|g_{21}|^2}\right)\right], \tag{13}$$

and if the channel satisfies the condition (1)*, the gap between outer bound and inner bound is at most* $c + 2$ *bits.*

Proof. The details are in Section IV. □

Theorem 2. *For the fading interference channel without feedback and the transmitters having no channel state information, the rate region described by* (14)−(20) *is achievable with* $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1\right)$*:*

$$R_1 \le \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 1 \tag{14}$$

$$R_2 \le \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 1 \tag{15}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] - 2 + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] \tag{16}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] - 2 + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] \tag{17}$$

$$R_1+R_2 \le \mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+|g_{21}|^2\right)\right]-2 + \mathbb{E}\left[\log\left(1+\lambda_{p2}|g_{22}|^2+|g_{12}|^2\right)\right] \quad (18)$$

$$2R_1+R_2 \le \mathbb{E}\left[\log\left(1+|g_{11}|^2+|g_{21}|^2\right)\right]-3 + \mathbb{E}\left[\log\left(1+\lambda_{p2}|g_{22}|^2+|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right] \quad (19)$$

$$R_1+2R_2 \le \mathbb{E}\left[\log\left(1+|g_{22}|^2+|g_{12}|^2\right)\right]-3 + \mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1+\lambda_{p2}|g_{22}|^2+\lambda_{p1}|g_{12}|^2\right)\right] \quad (20)$$

An outer bound for the non-feedback case is given by $(21)-(27)$.

$$R_1 \le \mathbb{E}\left[\log\left(1+|g_{11}|^2\right)\right] \quad (21)$$

$$R_2 \le \mathbb{E}\left[\log\left(1+|g_{22}|^2\right)\right] \quad (22)$$

$$R_1+R_2 \le \mathbb{E}\left[\log\left(1+|g_{22}|^2+|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right] \quad (23)$$

$$R_1+R_2 \le \mathbb{E}\left[\log\left(1+|g_{11}|^2+|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right] \quad (24)$$

$$R_1+R_2 \le \mathbb{E}\left[\log\left(1+|g_{21}|^2+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right] + \mathbb{E}\left[\log\left(1+|g_{12}|^2+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right] \quad (25)$$

$$2R_1+R_2 \le \mathbb{E}\left[\log\left(1+|g_{11}|^2+|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1+|g_{12}|^2+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right] + \mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right] \quad (26)$$

$$R_1+2R_2 \le \mathbb{E}\left[\log\left(1+|g_{22}|^2+|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1+|g_{21}|^2+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right] + \mathbb{E}\left[\log\left(1+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right], \quad (27)$$

and if the channel satisfies the condition (1)*, then the gap between outer bound and inner bound is at most* $c+1$ *bits.*

Proof. The details are in Section V. □

Our results can be applied for most of the well behaved continuous fading models by showing that they satisfy the condition (1). It turns out that the bursty interference model do not satisfy the condition (1) as we discuss in Section VI.

IV. The fading interference channel with feedback

For the inner bound we use the block Markov scheme from [7] which gives the following achievable region:

$$R_1 \le I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \quad (28)$$

$$R_1 \le I\left(U_1; Y_2, \underline{g_2}|U, X_2\right) + I\left(X_1; Y_1, \underline{g_1}|U_1, U_2, U\right) \quad (29)$$

$$R_2 \le I\left(U, U_1, X_2; Y_2, \underline{g_2}\right) \quad (30)$$

$$R_2 \le I\left(U_2; Y_1, \underline{g_1}|U, X_1\right) + I\left(X_2; Y_2, \underline{g_2}|U_1, U_2, U\right) \quad (31)$$

$$R_1+R_2 \le I\left(X_1; Y_1, \underline{g_1}|U_1, U_2, U\right) + I\left(U, U_1, X_2; Y_2, \underline{g_2}\right) \quad (32)$$

$$R_1+R_2 \le I\left(X_2; Y_2, \underline{g_2}|U_1, U_2, U\right) + I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \quad (33)$$

for all $p(u)\,p(u_1|u)\,p(u_2|u)\,p(x_1|u_1,u)\,p(x_2|u_2,u)$. Note that we have modified the expressions from [7] by replacing Y_1 with $\left(Y_1, \underline{g_1}\right)$ and Y_2 with $\left(Y_2, \underline{g_2}\right)$ to take the fading into account, since the receiver knows the received symbol itself and the channel strengths of the links. Now similar to [7] we choose the following Gaussian input distribution:

$$U \sim \mathcal{CN}\left(0, |\rho|^2\right), U_k \sim \mathcal{CN}\left(0, \lambda_{ck}\right), X_{pk} \sim \mathcal{CN}\left(0, \lambda_{pk}\right)$$

$$X_1 = e^{i\theta}U + U_1 + X_{p1}$$

$$X_2 = U + U_2 + X_{p2}$$

with $0 \le |\rho|^2 \le 1$, $0 \le \theta < 2\pi$,$\lambda_{ck}+\lambda_{pk} = 1-|\rho|^2$ and $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1-|\rho|^2\right)$. With this choice of λ_{pk} we perform the rate splitting according to the average inr in place of rate splitting based on the constant inr for static channels. Note that we have introduced an extra rotation θ for the first transmitter, which will become helpful in proving the capacity gap (see Appendix III). On evaluating the terms in $(28)-(33)$ for this choice of input distribution, we get the inner bound described by $(2)-(7)$, the calculations are deferred to Appendix I.

The outer bounds can be easily derived following the proof techniques from [7] using $\mathbb{E}\left[X_1X_2^*\right] = \rho$ while making changes relevant to the fading case. The calculations are deferred to the Appendix II.

Claim 3. The gap between the inner bound $(2)-(7)$ and the outer bound $(8)-(13)$ for the feedback case is atmost $c+2$ bits.

Proof. The condition we imposed on the fading distribution becomes important in proving a constant gap capacity result. We illustrate it by computing the gap δ_2 between the second inequality (9) of the outer bound and the second inequality (3) of the inner bound.

$$\delta_2 = \mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)|g_{12}|^2}\right)\right]$$

$$-\mathbb{E}\left[\log\left(1+\lambda_{p1}\left|g_{11}\right|^2+\lambda_{p2}\left|g_{21}\right|^2\right)\right]+2$$

$$\begin{aligned}
&\overset{(a)}{\leq} \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)INR_1+\left(1-|\rho|^2\right)\left|g_{11}\right|^2\right)\right]\\
&\quad-\mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)INR_1\right)\right]+c\\
&\quad\quad-\mathbb{E}\left[\log\left(1+\lambda_{p1}\left|g_{11}\right|^2+\lambda_{p2}\left|g_{21}\right|^2\right)\right]+2\\
&\leq \mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)\left|g_{11}\right|^2}{1+\left(1-|\rho|^2\right)INR_1}\right)\right]\\
&\quad-\mathbb{E}\left[\log\left(1+\lambda_{p1}\left|g_{11}\right|^2\right)\right]+2+c
\end{aligned}$$

where (a) follows from condition (1) on the distribution of $\left|g_{ij}\right|^2$ and Jensen's inequality. We have $\lambda_{p1}=\min\left(\frac{1}{INR_1},1-|\rho|^2\right)$; on considering the cases $\lambda_{p1}=1-|\rho|^2$ and $\lambda_{p1}=\frac{1}{INR_1}$ separately, it can be shown that

$$1+\frac{\left(1-|\rho|^2\right)\left|g_{11}\right|^2}{1+\left(1-|\rho|^2\right)INR_1}<1+\lambda_{p1}\left|g_{11}\right|^2.$$

Hence $\delta_2\leq c+2$ follows. More details on the computation of capacity gap for the feedback case are in Appendix III. ☐

V. The fading interference channel without feedback

From [2] we obtain that a Han-Kobayashi scheme for IC can achieve the following rate region for all $p\left(u_1\right)p\left(u_2\right)p\left(x_1|u_1\right)p\left(x_2|u_2\right)$. Note that we use $\left(Y_i,\underline{g_i}\right)$ instead of $\left(Y_i\right)$ in the actual result from [2] to account for the fading.

$$R_1\leq I\left(X_1;Y_1,\underline{g_1}|U_2\right) \tag{34}$$
$$R_2\leq I\left(X_2;Y_2,\underline{g_2}|U_1\right) \tag{35}$$
$$R_1+R_2\leq I\left(X_2,U_1;Y_2,\underline{g_2}\right)+I\left(X_1;Y_1,\underline{g_1}|U_1,U_2\right) \tag{36}$$
$$R_1+R_2\leq I\left(X_1,U_2;Y_1,\underline{g_1}\right)+I\left(X_2;Y_2,\underline{g_2}|U_1,U_2\right) \tag{37}$$
$$R_1+R_2\leq I\left(X_1,U_2;Y_1,\underline{g_1}|U_1\right)+I\left(X_2,U_1;Y_2,\underline{g_2}|U_2\right) \tag{38}$$
$$2R_1+R_2\leq I\left(X_1,U_2;Y_1,\underline{g_1}\right)+I\left(X_1;Y_1,\underline{g_1}|U_1,U_2\right)+I\left(X_2,U_1;Y_2,\underline{g_2}|U_2\right) \tag{39}$$
$$R_1+2R_2\leq I\left(X_2,U_1;Y_2,\underline{g_2}\right)+I\left(X_2;Y_2,\underline{g_2}|U_1,U_2\right)+I\left(X_1,U_2;Y_1,\underline{g_1}|U_1\right). \tag{40}$$

Now similar to that in [3], choose the Gaussian input distribution

$$U_k\sim\mathcal{CN}\left(0,\lambda_{ck}\right),\quad X_{pk}\sim\mathcal{CN}\left(0,\lambda_{pk}\right),\qquad k\in\{1,2\}$$
$$X_1=U_1+X_{p1}$$
$$X_2=U_2+X_{p2}$$

where $\lambda_{ck}+\lambda_{pk}=1$ and $\lambda_{pk}=\min\left(\frac{1}{INR_k},1\right)$. Here we introduced the rate splitting using the average inr. On evaluating the region described by $(34)-(40)$ with this choice of input distribution, we get the region described by $(14)-(20)$; the computations are similar to that of the feedback case. The outer bounds easily follow from the results in [3] by suitably modifying for the fading case, and by using the outer bounds (12) and (13) for feedback case after setting $\mathbb{E}\left[X_1X_2^*\right]=\rho=0$.

Claim 4. The gap between the inner bound $(14)-(20)$ and the outer bound $(21)-(27)$ for the feedback case is atmost $c+1$ bits.

Proof. The proof for the capacity gap again uses the condition (1) on the fading distribution. More details are given in Appendix IV. ☐

VI. Fading models

Here we discuss the fading models that satisfy the required condition (1). We first note that $\phi(a)=\log\left(a+\mu_W\right)-\mathbb{E}\left[\log\left(a+W\right)\right]\geq0$ due to Jensen's inequality, so the quantity we are interested in is a Jensen's gap. We will simplify the condition further. On taking the derivative with respect to a and again using Jensen's inequality we get

$$(\ln 2)\,\phi'(a)=\frac{1}{a+\mu_W}-\mathbb{E}\left[\frac{1}{a+W}\right]\leq0.$$

Hence $\phi(a)$ achieves the maximum value at $a=0$ in the range $[0,\infty)$. Hence we just have to show

$$\log\left(\mu_W\right)-\mathbb{E}\left[\log\left(W\right)\right]\leq c.$$

Note that if the distribution gets scaled by a factor k, then we have the same gap

$$\log\left(k\mu_W\right)-\mathbb{E}\left[\log\left(kW\right)\right]=\log\left(\mu_W\right)-\mathbb{E}\left[\log\left(W\right)\right],$$

and hence we can fix the scaling by requiring the mean of W to be 1 . So let $\mu_W=1$, then the required condition reduces to

$$\mathbb{E}\left[\log\left(W\right)\right]\geq-c. \tag{41}$$

Hence it follows that for any distribution that has a point mass at 0, we cannot guarantee a constant capacity gap, since it has $\mathbb{E}\left[\log\left(W\right)\right]=-\infty$. Now we discuss a few distributions that can be easily shown to satisfy the required condition for the scheme.

A. Gamma distribution

The probability density function for Gamma distribution is given by

$$f\left(w\right)=\frac{w^{k-1}e^{-\frac{w}{\theta}}}{\theta^k\Gamma(k)}$$

for $w>0$ and $k,\theta>0$. We use the sufficient condition (41) here. So we need to bound $\mathbb{E}\left[\log\left(W\right)\right]$ when $\mathbb{E}\left[W\right]=1$. It is known for the Gamma distribution that $\mathbb{E}\left[W\right]=k\theta$

and $\mathbb{E}[\ln(W)] = \psi(k) + \ln(\theta)$, where ψ is the digamma function. Since $\mathbb{E}[W] = 1$ we get $\theta = \frac{1}{k}$ and hence

$$\begin{aligned}\mathbb{E}[\ln(W)] &= \psi(k) + \ln(\theta) \\ &= \psi(k) - \ln(k).\end{aligned}$$

If it is specified that $0 < \alpha \le k$, we first use the following property of digamma function

$$\psi(k) = \psi(k+1) - \frac{1}{k},$$

and then use the inequality from [1]

$$\ln\left(k + \frac{1}{2}\right) < \psi(k+1) < \ln\left(k + e^{-\gamma}\right).$$

Hence

$$\begin{aligned}\ln(2)\,\mathbb{E}[\log(W)] &> \ln\left(k + \frac{1}{2}\right) - \ln(k) - \frac{1}{k} \\ &= \ln\left(1 + \frac{1}{2k}\right) - \frac{1}{k} \\ \mathbb{E}[\log(W)] &> \log\left(1 + \frac{1}{2\alpha}\right) - \frac{1}{\alpha \ln(2)}. \qquad (42)\end{aligned}$$

The last steps follows because the function involved is decreasing in k in the range $(0, \infty)$.

1) Rayleigh fading: In Rayleigh fading model the $|g_{ij}|^2$ is exponentially distributed with mean INR_i. The exponential distribution itself is a special case of Gamma distribution with $k = 1$. Substituting $\alpha = 1$ in (42) we get

$$\mathbb{E}[\log(W)] > -.86 = -c.$$

Using the above value of c, we get the feedback capacity gap as $c + 2 = 2.86$ bits, and for the non feedback case we get a gap of $c + 1 = 1.86$ bits.

Also the distributions for the Nakagami fading can be obtained as special cases of the Gamma distribution; then the capacity gap will depend upon the parameters used in the model.

B. Weibull distribution

The probability density function for Weibull distribution is given by

$$f(w) = \frac{k}{\lambda}\left(\frac{w}{\lambda}\right)^{k-1} e^{-(w/\lambda)^k}$$

for $x > 0$ with $k, \lambda > 0$. Here $\mathbb{E}[W] = \lambda\Gamma\left(1 + \frac{1}{k}\right)$ and $\mathbb{E}[\ln(W)] = \ln(\lambda) - \frac{\gamma}{k}$, where $\Gamma(w)$ denotes the gamma function and γ is the Euler's constant. By setting the mean to be unity, we get $\lambda = \frac{1}{\Gamma\left(1+\frac{1}{k}\right)}$. Hence if it is specified that $0 < \alpha \le k$, then we get

$$\mathbb{E}[\log(W)] \ge -\log\left(\Gamma\left(1 + \frac{1}{\alpha}\right)\right) - \frac{\gamma}{\alpha \ln(2)}.$$

Note that exponential distribution can be obtained from Weibull distribution also by setting $k = 1$. We get $\mathbb{E}[\log(W)] = -\frac{\gamma}{\ln(2)}$ for the Rayleigh fading case. This gives the capacity gap computed more accurately for the Rayleigh fading case as 2.83 bits for the feedback case, and 1.83 bits for the non feedback case.

C. Other distributions

Here we give a lemma that can be used to verify whether for a given fading model, our scheme of rate-splitting at average inr yield a constant gap capacity result.

Lemma 5. *If the cumulative distribution function $F(w)$ of W is such that $F(w) \le aw^b$ for $w \in [0, \epsilon]$ where $0 \le a, 0 < b, 0 < \epsilon \le 1$, then*

$$\mathbb{E}[\ln(W)] \ge \ln(\epsilon) + a\epsilon^b \ln(\epsilon) - \frac{a\epsilon^b}{b}.$$

Proof. The condition in this lemma ensures that the probability density function $f(w)$ grows slow enough as $w \to 0^-$ so that $f(w)\ln(w)$ is integrable at 0. Also the behaviour for large values of w is not relevant here, since we are looking for a lower bound on $\mathbb{E}[\ln(W)]$. The detailed proof is given in Appendix V. □

Hence if the fading model guarantees $F(w) \le aw^b$ for $w \in [0, \epsilon]$ with some $0 \le a, 0 < b, 0 < \epsilon \le 1$ for the scaled version of W with unit mean, then we can find a c satisfying the required condition (1).

VII. Conclusion

We proved that the rate-splitting schemes for the static 2-user Gaussian IC without CSIT [3], [2] and that with delayed feedback [7] , can be extended to the fading case for a class of fading distributions. The proof for optimality to within a constant gap, relies on the sufficient condition, which the fading distribution is assumed to satisfy. Our technique does not work for the bursty interference case since it does not satisfy the condition. It would be interesting to study if bursty interference scheme [10] and our proposed scheme can be combined to tackle arbitrary fading distributions.

VIII. Acknowledgements

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Appendix I
Proof of achievability for feedback case

We evaluate the term in the first inner bound inequality (28) . The other terms can be similarly evaluated.

$$\begin{aligned}I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) &\overset{(a)}{=} I\left(U, U_2, X_1; Y_1|\underline{g_1}\right) \\ &= h\left(Y_1|\underline{g_1}\right) - h\left(Y_1|\underline{g_1}, U, U_2, X_1\right),\end{aligned}$$

$$\begin{aligned}&\text{variance}\left(Y_1|\underline{g_1}\right) \\ &= \text{variance}\left(g_{11}X_1 + g_{21}X_2 + Z_1|g_{11}, g_{21}\right) \\ &= |g_{11}|^2 + |g_{21}|^2 + g_{11}^* g_{21}\mathbb{E}[X_1^* X_2] + g_{11}g_{21}^*\mathbb{E}[X_1 X_2^*] \\ &\quad + 1 \\ &= |g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \,\text{Re}\left(g_{11}g_{21}^* e^{i\theta}\right) + 1\end{aligned}$$

$$
\begin{aligned}
&h\left(Y_1|\underline{g_1},U,U_2,X_1\right)\\
&\quad= h\left(g_{11}X_1+g_{21}X_2+Z_1|\underline{g_1},U,U_2,X_1\right)\\
&\quad= h\left(g_{21}X_{p2}+Z_1|\underline{g_1}\right)\\
&\quad= \mathbb{E}\left[\log\left(1+\lambda_{p2}\left|g_{21}\right|^2\right)\right]+\log\left(2\pi e\right)\\
&\quad\overset{(b)}{\leq} \mathbb{E}\left[\log\left(1+\frac{1}{INR_2}\left|g_{21}\right|^2\right)\right]+\log\left(2\pi e\right)\\
&\quad\overset{(c)}{\leq} \log\left(2\right)+\log\left(2\pi e\right)\\
&\quad= 1+\log\left(2\pi e\right),\\
&\therefore I\left(U,U_2,X_1;Y_1,\underline{g_1}\right)\\
&\quad\geq \mathbb{E}\left[\log\left(\left|g_{11}\right|^2+\left|g_{21}\right|^2+2\left|\rho\right|^2\mathrm{Re}\left(g_{11}g_{21}^*e^{i\theta}\right)+1\right)\right]\\
&\quad-1
\end{aligned}
$$

where (a) uses independence, (b) uses the monotonicity of expectation and $\lambda_{pi}\leq\frac{1}{INR_i}$, and (c) follows from Jensen's inequality.

Appendix II
Proof of outerbounds for feedback case

Following the Suh-Tse methods [7], we let $\mathbb{E}\left[X_1X_2^*\right]=\rho$. We use the notation $\underline{g_1}=\left[g_{11},g_{21}\right]$, $\underline{g_2}=\left[g_{22},g_{12}\right]$, $\underline{g}=\left[g_{11},g_{21},g_{22},g_{12}\right]$, $S_1=g_{12}X_1+Z_2$, and $S_2=g_{21}X_2+Z_1$. We let $\mathbb{E}\left[X_1X_2^*\right]=\rho=\left|\rho\right|e^{i\theta}$. On choosing a uniform distribution of messages we get

$$
\begin{aligned}
&n(R_1-\epsilon_n)\\
&\quad\overset{(a)}{\leq} I\left(W_1;Y_1^n|\underline{g_1^n}\right)\\
&\quad\overset{(b)}{\leq} \sum\left(h\left(Y_{1i}|\underline{g_{1i}}\right)-h\left(Z_{1i}\right)\right)\\
&\quad= \sum\left(\mathbb{E}_{\underline{\tilde{g}_{1i}}}\left[h\left(Y_{1i}|\underline{g_{1i}}=\underline{\tilde{g}_{1i}}\right)-h\left(Z_{1i}\right)\right]\right)\\
&\quad\overset{(c)}{=} \mathbb{E}_{\underline{\tilde{g}_1}}\left[\sum\left(h\left(Y_{1i}|\underline{g_{1i}}=\underline{\tilde{g}_1}\right)-h\left(Z_{1i}\right)\right)\right]
\end{aligned}
$$

$$
R_1\leq\mathbb{E}\left[\log\left(\left|g_{11}\right|^2+\left|g_{21}\right|^2+\left(\rho^*g_{11}^*g_{21}+\rho g_{11}g_{21}^*\right)+1\right)\right]
$$

where (a) follows from Fano's inequality, (b) follows from the fact that conditioning reduces entropy, and (c) follows from the fact that $\underline{\tilde{g}_{1i}}$ are i.i.d. Now we bound R_1 in a second way as done in [7]:

$$
\begin{aligned}
&n(R_1-\epsilon_n)\\
&\quad\leq I\left(W_1;Y_1^n,\underline{g_1^n}\right)\\
&\quad\leq I\left(W_1;Y_1^n,\underline{g_1^n},Y_2^n,\underline{g_2^n},W_2\right)\\
&\quad= I\left(W_1;\underline{g_1^n},\underline{g_2^n},W_2\right)\\
&\qquad+I\left(W_1;Y_1^n,Y_2^n|\underline{g_1^n},\underline{g_2^n},W_2\right)\\
&\quad= 0+I\left(W_1;Y_1^n,Y_2^n|\underline{g}^n,W_2\right)\\
&\quad= h\left(Y_1^n,Y_2^n|\underline{g}^n,W_2\right)-h\left(Y_1^n,Y_2^n|\underline{g}^n,W_1,W_2\right)\\
&\quad= \sum\left[h\left(Y_{1i},Y_{2i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i-1}\right)\right]\\
&\qquad-\sum\left[h\left(Z_{1i}\right)+h\left(Z_{2i}\right)\right]\\
&\quad= \sum\left[h\left(Y_{2i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i-1}\right)\right]\\
&\qquad+\sum\left[h\left(Y_{1i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i}\right)\right]\\
&\qquad-\sum\left[h\left(Z_{1i}\right)+h\left(Z_{2i}\right)\right]\\
&\quad\overset{(a)}{=} \sum\left[h\left(Y_{2i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i-1},X_2^i\right)\right]\\
&\qquad+\sum\left[h\left(Y_{1i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i},S_{1i},X_2^i\right)\right]\\
&\qquad-\sum\left[h\left(Z_{1i}\right)+h\left(Z_{2i}\right)\right]\\
&\quad\overset{(b)}{\leq} \sum\left[h\left(Y_{2i}|\underline{g_i},X_{2i}\right)-h\left(Z_{2i}\right)\right]\\
&\qquad+\sum\left[h\left(Y_{1i}|\underline{g_i},S_{1i},X_{2i}\right)-h\left(Z_{1i}\right)\right]\\
&\quad\overset{(c)}{=} \mathbb{E}_{\underline{\tilde{g}}}\left[\sum\left(h\left(Y_{2i}|X_{2i},\underline{g_i}=\underline{\tilde{g}}\right)-h\left(Z_{2i}\right)\right)\right]\\
&\qquad+\mathbb{E}_{\underline{\tilde{g}}}\left[\sum\left(h\left(Y_{1i}|S_{1i},X_{2i},\underline{g_i}=\underline{\tilde{g}}\right)-h\left(Z_{1i}\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
R_1&\overset{(d)}{\leq}\mathbb{E}\left[\log\left(1+\left(1-\left|\rho\right|^2\right)\left|g_{12}\right|^2\right)\right]\\
&\quad+\mathbb{E}\left[\log\left(1+\frac{\left(1-\left|\rho\right|^2\right)\left|g_{11}\right|^2}{1+\left(1-\left|\rho\right|^2\right)\left|g_{12}\right|^2}\right)\right]
\end{aligned}
$$

where (a) follows from the fact that X_2^i is a function of $\left(W_2,Y_2^{i-1},\underline{g}^{i-1}\right)$ and S_1^i is a function of $\left(Y_2^i,X_2^i\right)$, (b) follows from the fact that conditioning reduces entropy, (c) follows from the fact that $\underline{\tilde{g}_i}$ are i.i.d., and (d) follows from the Suh-Tse results [7]. The other outer bounds can be derived similarly following [7] and making suitable changes to account for fading as we illustrated in the previous two derivations.

Appendix III
Proof of the capacity gap for feedback case

We compare the corresponding equations in outer and inner bounds. Denote the gap between the first outer bound and inner bound by δ_1, for the second pair denote the gap by δ_2, and so on. Choose θ in the inner bound to match $\arg\left(\rho\right)$ in the outer bound. We get

$$
\begin{aligned}
\delta_1&=\mathbb{E}\left[\log\left(\left|g_{11}\right|^2+\left|g_{21}\right|^2+2\left|\rho\right|\mathrm{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)+1\right)\right]\\
&\quad-\mathbb{E}\left[\log\left(\left|g_{11}\right|^2+\left|g_{21}\right|^2+2\left|\rho\right|^2\mathrm{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)+1\right)\right]\\
&\quad+1\\
&=\mathbb{E}\left[\log\left(\frac{1+\left|g_{11}\right|^2+\left|g_{21}\right|^2+2\left|\rho\right|\mathrm{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{1+\left|g_{11}\right|^2+\left|g_{21}\right|^2+2\left|\rho\right|^2\mathrm{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}\right)\right]\\
&\quad+1\\
&=\mathbb{E}\left[\log\left(\frac{1+\frac{1}{\left|g_{11}\right|^2+\left|g_{21}\right|^2}+\left|\rho\right|\left(\frac{2\mathrm{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{\left|g_{11}\right|^2+\left|g_{21}\right|^2}\right)}{1+\frac{1}{\left|g_{11}\right|^2+\left|g_{21}\right|^2}+\left|\rho\right|^2\left(\frac{2\mathrm{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{\left|g_{11}\right|^2+\left|g_{21}\right|^2}\right)}\right)\right]\\
&\quad+1.
\end{aligned}
$$

We have

$$\left|\frac{2\text{Re}\left(e^{i\theta}g_{11}g_{21}^{*}\right)}{|g_{11}|^2+|g_{21}|^2}\right| = \frac{\left|e^{-i\theta}g_{11}^{*}g_{21}+e^{i\theta}g_{11}g_{21}^{*}\right|}{|g_{11}|^2+|g_{21}|^2} \leq 1,$$

hence we call $\frac{e^{-i\theta}g_{11}^{*}g_{21}+e^{i\theta}g_{11}g_{21}^{*}}{|g_{11}|^2+|g_{21}|^2} = \sin\phi$ and let

$$|g_{11}|^2+|g_{21}|^2 = r^2.$$

Therefore

$$\delta_1 = \mathbb{E}\left[\log\left(\frac{1+\frac{1}{r^2}+|\rho|\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi}\right)\right]+1.$$

If $\sin\phi<0$, then

$$\frac{1+\frac{1}{r^2}+|\rho|\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi} \leq 1.$$

If $\sin\phi>0$, then

$$\begin{aligned}\frac{1+\frac{1}{r^2}+|\rho|\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi} &= 1+\frac{\left(|\rho|-|\rho|^2\right)\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi}\\ &\leq 2\end{aligned}$$

since $0 \leq \left(|\rho|-|\rho|^2\right)\sin\phi \leq 1$ and $1+\frac{1}{r^2}+|\rho|^2\sin\phi > 1$. Hence

$$\delta_1 \leq 2.$$

We have already shown in Section IV that $\delta_2 \leq 2+c$. By inspection of the other bounding inequalities, it is clear that the capacity gap is at-most $2+c$ bits.

APPENDIX IV
PROOF OF CAPACITY GAP FOR NON-FEEDBACK CASE

Denote the gap between the first outer bound (21) and first inner bound (14) by δ_1, δ_2 for the second pair and so on. Clearly

$$\begin{aligned}\delta_1 &\leq 1\\ \delta_2 &\leq 1.\end{aligned}$$

Now

$$\begin{aligned}\delta_3 &= 2+\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right]\\ &\quad -\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]\\ &\overset{(a)}{\leq} 2+\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+INR_1}\right)\right]+c\\ &\quad -\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2\right)\right].\end{aligned}$$

The step (a) follows from Jensen's inequality and condition (1) on $|g_{12}|^2$. We have $\lambda_{p1} = \min\left(\frac{1}{INR_1},1\right) \geq \frac{1}{INR_1+1}$, hence

$$\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+INR_1}\right)\right]-\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2\right)\right] \leq 0.$$

Therefore

$$\delta_3 \leq 2+c.$$

Similarly

$$\begin{aligned}\delta_4 &\leq 2+c\\ \delta_5 &\leq 2+2c\\ \delta_6 &\leq 3+2c\\ \delta_7 &\leq 3+2c.\end{aligned}$$

For δ_5,δ_6, and δ_7 we have to use the condition (1) twice and hence $2c$ appears. Hence it easily follows that the capacity gap is at-most $c+1$ bits.

APPENDIX V
PROOF OF LEMMA 5

Proof. We have $F(w) \leq aw^b$ for $w \in [0,\epsilon]$ where $a \geq 0, b>0, 1 \geq \epsilon > 1$. Now using integration by parts we get

$$\begin{aligned}\mathbb{E}[\ln(W)] &\geq \int_0^1 f(w)\ln(w)\\ &= \int_0^\epsilon f(w)\ln(w)+\int_\epsilon^1 f(w)\ln(w)\\ &= [F(w)\ln(w)]_0^\epsilon - \int_0^\epsilon F(w)\frac{1}{w}\\ &\quad + \int_\epsilon^1 f(w)\ln(w)\\ &\geq \left[aw^b\ln(w)\right]_0^\epsilon - \int_0^\epsilon aw^b\frac{1}{w}\\ &\geq a\epsilon^b\ln(\epsilon)-\frac{a\epsilon^b}{b}+\ln(\epsilon).\end{aligned}$$

Note that $\ln(w)$ is negative in the range $[0,1)$, thus we get the desired inequalities in the previous steps. □

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