Approximate Capacity of Fast Fading Interference Channels with no CSIT

Joyson Sebastian, Can Karakus, Suhas Diggavi

Abstract

We develop a characterization of fading models, which assigns a number called *logarithmic Jensen's gap* to a given fading model. We show that as a consequence of a finite logarithmic Jensen's gap, approximate capacity region can be obtained for fast fading interference channels (FF-IC) for several scenarios. We illustrate three instances where a constant capacity gap can be obtained as a function of the logarithmic Jensen's gap. Firstly for an FF-IC with neither feedback nor channel state information at transmitter (CSIT), if the fading distribution has finite logarithmic Jensen's gap, we show that a rate-splitting scheme based on average interference-to-noise ratio (inr) can achieve its approximate capacity. Secondly we show that a similar scheme can achieve the approximate capacity of FF-IC with feedback and delayed CSIT, if the fading distribution has finite logarithmic Jensen's gap. Thirdly, when this condition holds, we show that point-to-point codes can achieve approximate capacity for a class of FF-IC with feedback. We prove that the logarithmic Jensen's gap is finite for common fading models, including Rayleigh and Nakagami fading, thereby obtaining the approximate capacity region of FF-IC with these fading models.

I. Introduction

The 2-user Gaussian IC is a simple model that captures the effect of interference in wireless networks. Significant progress has been made in the last decade in understanding the capacity of the Gaussian IC [10], [6], [7], [21]. In practice the links in the channel could be time-varying rather than static. Characterizing the capacity of FF-IC without CSIT has been an open problem. In this paper we make progress in this direction by obtaining the capacity region of certain classes of FF-IC without CSIT within a constant bits/s/Hz.

Shorter versions of this work appeared in [20], [19] with outline of proofs. This version has complete proofs. This work was supported in part by NSF grants 1514531 and 1314937.

A. Related work

Previous works have characterized the capacity region to within a constant gap for the IC where the channel is known at the transmitter and receiver. The capacity region of the 2-user IC without feedback was characterized to within 1 bit/s/Hz in [7]. In [21], Suh and Tse characterized the capacity region of the IC with feedback to within 2 bits/s/Hz. These results were based on the Han-Kobayashi scheme [10], where the transmitters use superposition coding splitting their messages into common and private parts, and the receivers use joint decoding. Other variants of wireless networks based on the IC model have been studied in literature. The interference relay channel (IRC), which is obtained by adding a relay to the 2-user interference channel (IC) setup, was introduced in [17] and was further studied in [23], [14], [3], [9]. In [28], Wang and Tse studied the IC with receiver cooperation. The IC with source cooperation was studied in [16], [29].

When the channels are time varying most of the existing techniques for IC cannot be used without CSIT. In [8], Farsani showed that if each transmitter of FF-IC has knowledge of the inr to the non-corresponding receiver[1], the capacity region can be achieved within 1 bit/s/Hz. The idea of interference alignment [5] has been extended to FF-IC to obtain the degrees of freedom (DoF) region for certain cases. The degrees of freedom region for the MIMO interference channel with delayed CSIT was studied in [26]. Their results show that when *all users have single antenna*, the DoF region is same for the cases of no CSIT, delayed CSIT and instantaneous CSIT. The results from [22] show that DoF region for FF-IC with output feedback and delayed CSIT is contained in the DoF region for the case with instantaneous CSIT and no feedback. Kang and Choi [11] considered interference alignment for the K-user FF-IC with delayed channel state feedback and showed a result of $\frac{2K}{K+2}$ DoF. They also showed the same DoF can be achieved using a scaled output feedback, but without channel state feedback. Therefore, the above works have characterizations for DoF for several fading scenrios, and also show that for single antenna systems, feedback is not very effective in terms of DoF. However, as we show in this paper, the situation changes when we look for more than DoF, and for approximate optimality of the entire capacity region. In particular, we allow for arbitrary channel gains, and do not limit ourselves to

[1]For Tx1 the non-corresponding receiver is Rx2 and similarly for Tx2 the non-corresponding receiver is Rx1

SNR-scaling results[2]. In particular, we show that though the capacity region is same (within a constant) for the cases of no CSIT, delayed CSIT and instantaneous CSIT, there is a significant difference with output feedback. When there is output feedback and delayed CSIT the capacity region is larger than that for the case with with no feedback and instantaneous CSIT in contrast to the DoF result from [22]. This gives us a finer understanding of the role of CSIT as well as feedback in FF-IC with arbitary (and potentially asymmetric) link strengths, and is one of the main contributions of this paper.

Some simplified fading models have been introduced to characterize the capacity region of the FF-IC in the absence of CSIT. In [27], Wang *et al.* considered the bursty IC, where the presence of interference is governed by a Bernoulli random state. The capacity of one-sided IC under ergodic layered erasure model, which considers the channel as a time-varying version of the binary expansion deterministic model [2], was studied in [1], [30]. The binary fading IC, where the channel gains, the transmit signals and the received signals are in the binary field was studied in [24], [25] by Vahid *et al.* In spite of these efforts, the capacity region of FF-IC without CSIT is still unknown, and this paper presents what we believe to be the first approximate characterization of the capacity region of FF-IC without CSIT, for a class of fading models satisfying the regularity condition, defined as the finite *logarithmic Jensen's gap*.

B. Contribution and outline

In this paper we first introduce the notion of *logarithmic Jensen's gap* for fading models. This is defined in Section III as a number calculated for a fading model depending on the probability distribution for the channel strengths. It is effectively the supremum of $\log\left(\mathbb{E}\left[\text{link strength}\right]\right) - \mathbb{E}\left[\log\left(\text{link strength}\right)\right]$ over all links and operating regimes of the system. We show that common fading models including Rayleigh and Nakagami fading have finite logarithmic Jensen's gap, but some fading models (like bursty fading [27]) have infinite logarithmic Jensen's gap. Subsequently we show the usefulness of logarithmic Jensen's gap in obtaining approximate capacity regions of FF-ICs without CSIT. We show that Han-Kobayashi type rate-splitting schemes [10], [6], [7], [21] based on inr, when extended to rate-splitting schemes based on $\mathbb{E}\left[inr\right]$ for the FF-ICs, give the capacity gap as a function of logarithmic Jensen's gap, yielding the approximate

[2]However, we can also use our results to get the *generalized* DoF studied in [7] for the FF-IC. This shows that for generalized DoF, feedback indeed helps, as shown in our results.

capacity characterization for fading models that have finite logarithmic Jensen's gap. Since our rate-splitting is based on $\mathbb{E}[inr]$, it does not need CSIT. The constant gap capacity result is first obtained for FF-IC without feedback or CSIT. We also show that for the FF-IC without feedback, instantaneous CSIT cannot improve the capacity region over the case with no CSIT, except for a constant gap. We subsequently study FF-IC with feedback and delayed CSIT to obtain a constant gap capacity result. In this case, having instantaneous CSIT cannot improve the capacity region over the case with delayed CSIT.

The usefulness of logarithmic Jensen's gap is further illustrated by analyzing a scheme based on point-to-point codes for a class of FF-IC with feedback, where we again obtain capacity gap as a function of logarithmic Jensen's gap. Our scheme is based on amplify-and-forward relaying, similar to the one proposed in [21]. It effectively induces a 2-tap inter-symbol-interference (ISI) channel for one of the users and a point-to-point feedback channel for the other user. The work in [21] had similarly shown that an amplify-and-forward based feedback scheme can achieve the symmetric rate point, without using rate-splitting. Our scheme can be considered as an extension to this scheme, which enables us to approximately achieve the entire capacity region. Our analysis also yields a capacity bound for a 2-tap fading ISI channel, the tightness of the bound again determined by the logarithmic Jensen's gap.

The paper is organized as follows. In section II we describe the system setup and the notations. In section III we develop the logarithmic Jensen's gap characterization for fading models. We illustrate a few applications of logarithmic Jensen's gap characterization in the later sections: in section IV, by obtaining approximate capacity region of FF-IC without feedback, in section V, by obtaining approximate capacity region of FF-IC with feedback and delayed CSIT, and in section VI, by developing point-to-point codes for a class of FF-IC with feedback.

II. Model and Notation

We consider the two-user FF-IC

$$Y_1(l) = g_{11}(l)X_1(l) + g_{21}(l)X_2(l) + Z_1(l) \tag{1}$$

$$Y_2(l) = g_{12}(l)X_1(l) + g_{22}(l)X_2(l) + Z_2(l) \tag{2}$$

where $Y_i(l)$ is the channel output of receiver i (Rxi) at time l, $X_i(l)$ is the input of transmitter i (Txi) at time l, $Z_i(l) \sim \mathcal{CN}(0,1)$ is complex AWGN noise process at Rxi, and $g_{ij}(l)$ is the time-variant random channel gain. The channel gain processes $\{g_{ij}(l)\}$ are independent across links (i, j) as well as over time. The transmitters are assumed to have no knowledge of the channel gain realizations, but the receivers do have full knowledge of their corresponding channels. We assume that the phase of $g_{ij}(l)$ is uniformly distributed in $[0, 2\pi]$, $|g_{ij}(l)|^2$ is distributed according to ϕ_{ij}, $i, j \in \{1, 2\}$, and $\Phi := \{\phi_{ij}\}_{i,j\in\{1,2\}}$. We assume average power constraint $\frac{1}{n}\sum_{l=1}^{n}|X_i(l)|^2 \leq 1, i = 1, 2$ at the transmitters, and assume Txi has a message $W_i \in \{1, ., 2^{NR_i}\}$, for a block length of N, intended for Rxi for $i = 1, 2$, and W_1, W_2 are independent of each other. We denote $SNR_i := \mathbb{E}\left[|g_{ii}|^2\right]$ for $i = 1, 2$, and $INR_i := \mathbb{E}\left[|g_{ij}|^2\right]$ for $i \neq j$. For the instantaneous interference channel gains we use $inr_i := |g_{ij}|^2$, $i \neq j$. Note that we allow for arbitrary channel gains, and do not limit ourselves to SNR-scaling results, but get an approximate characterization of the FF-IC capacity region.

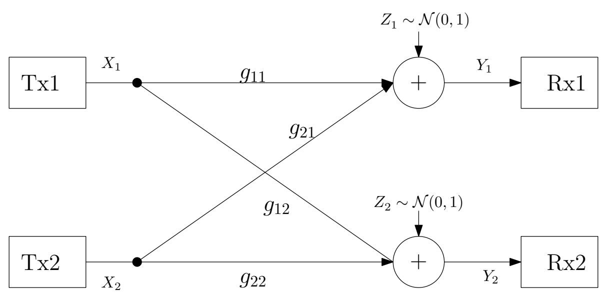

Figure 1. The channel model without feedback.

Under the feedback model (Figure 2), after each reception, each receiver reliably feeds back the received symbol and the channel states to its corresponding transmitter. For example, at time l, Tx1 receives $(Y_1(l-1), g_{11}(l-1), g_{21}(l-1))$ from Rx1. Thus $X_1(l)$ is allowed to be a function of $\left(W_1, \{Y_1(k), g_{11}(k), g_{21}(k)\}_{k<l}\right)$.

We define symmetric FF-IC to be a FF-IC such that $g_{11} \sim g_{22} \sim g_d$ and $g_{12} \sim g_{21} \sim g_c$, all of them still being independent. Here g_d and g_c are dummy random variables according to which the direct links and cross links are distributed. We denote $SNR := \mathbb{E}\left[|g_d|^2\right]$, and $INR := \mathbb{E}\left[|g_c|^2\right]$, for the symmetric case.

We use the vector notation $\underline{g}_1 = [g_{11}, g_{21}]$, $\underline{g}_2 = [g_{22}, g_{12}]$ and $\underline{g} = [g_{11}, g_{21}, g_{22}, g_{12}]$. For schemes involving multiple blocks (phases) we use the notation $X_k^{(i)N}$, where k is the user index, i is the block (phase) index and N is the number of symbols per block. The notation $X_k^{(i)}(j)$ indicates the j^{th} symbol in the i^{th} block (phase) of k^{th} user. We explain this in Figure 3.

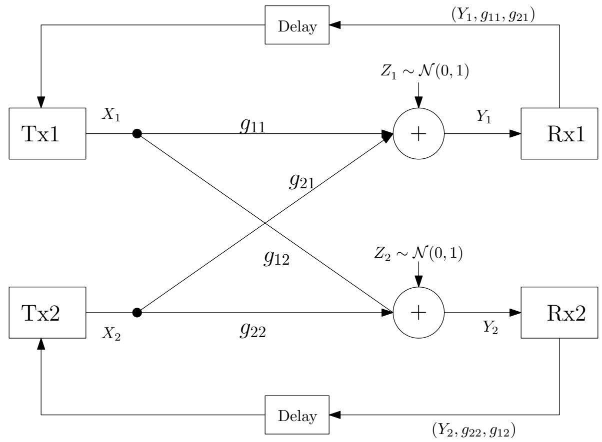

Figure 2. The channel model with feedback.

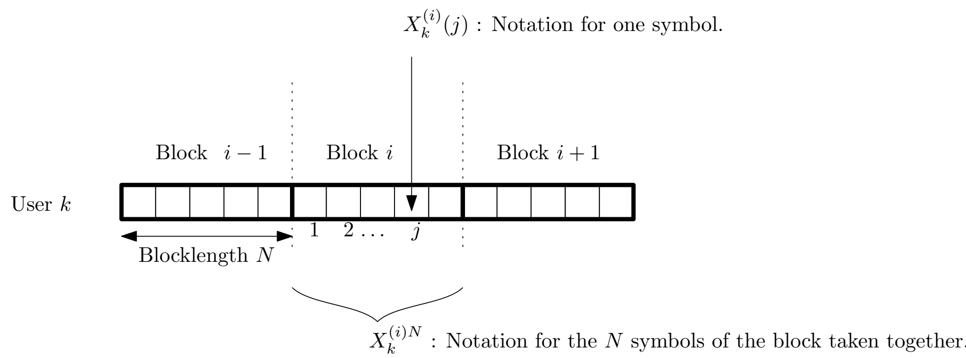

Figure 3. The notation for schemes involving multiple blocks (phases).

The natural logarithm is denoted by $\ln()$ and the logarithm with base 2 is denoted by $\log()$. Also we define $\log^+(\cdot) := \max(\log(\cdot), 0)$. For obtaining approximate capacity region of ICs, we say that a rate region $\mathcal{R}$ achieves a capacity gap of δ if for any $(R_1, R_2) \in \mathcal{C}$, $(R_1 - \delta, R_2 - \delta) \in \mathcal{R}$, where $\mathcal{C}$ is the capacity region of the channel.

III. A LOGARITHMIC JENSEN'S GAP CHARACTERIZATION FOR FADING MODELS

Definition 1. For a given fading model, let $\Phi = \left\{\phi : |g_{ij}|^2 \sim \phi, \text{ for some } i, j \in \{1, 2\}\right\}$ be the set of all probability distributions, that the fading model induce on the channel link strengths $|g_{ij}|^2$, across all operating regimes of the system. We define logarithmic Jensen's gap c_{JG} of the fading model to be

$$c_{JG} = \sup_{a \in \mathbb{R}^+, W \sim \phi \in \Phi} \left(\log\left(a + \mathbb{E}\left[W\right]\right) - \mathbb{E}\left[\log\left(a + W\right)\right]\right). \quad (3)$$

In other words it is the smallest value of c such that

$$\log\left(a+\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(a+W\right)\right]\leq c, \tag{4}$$

for any $a \geq 0$, for *any* $\phi \in \Phi$, with W distributed according to ϕ.

The following lemma converts requirement in Definition 1 to a simpler form.

Lemma 2. *The requirement* $\log\left(a+\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(a+W\right)\right]\leq c$ *for any* $a\geq 0$*, is equivalent to* $\log\left(\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(W\right)\right]=-\mathbb{E}\left[\log\left(W'\right)\right]\leq c$*, where* $W'=\frac{W}{\mathbb{E}[W]}$.

Proof: We first note that letting $a=0$ in the requirement $\log\left(a+\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(a+W\right)\right]\leq c$ shows that $\log\left(\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(W\right)\right]=\mathbb{E}\left[\log\left(W'\right)\right]\geq -c$ is necessary.

To prove the converse, note that $\xi\left(a\right)=\log\left(a+\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(a+W\right)\right]\geq 0$ due to Jensen's inequality. Taking derivative with respect to a and again using Jensen's inequality we get

$$\left(\ln 2\right)\xi'\left(a\right)=\frac{1}{a+\mathbb{E}\left[W\right]}-\mathbb{E}\left[\frac{1}{a+W}\right]\leq 0. \tag{5}$$

Hence $\xi\left(a\right)$ achieves the maximum value at $a=0$ in the range $[0,\infty)$. Hence we have the equivalent condition

$$\log\left(\mathbb{E}\left[W\right]\right)-\mathbb{E}\left[\log\left(W\right)\right]\leq c, \tag{6}$$

which is equivalent to

$$-\mathbb{E}\left[\log\left(W'\right)\right]\leq c. \tag{7}$$

■

Hence it follows that for any distribution that has a point mass at 0 (for example, bursty interference model [27]), we do not have a finite logarithmic Jensen's gap, since it has $\mathbb{E}\left[\log\left(W'\right)\right]=-\infty$. Now we discuss a few distributions that can be easily shown to have a finite logarithmic Jensen's gap. Note that any finite c, which satisfies Equation (4), is an upper bound to the logarithmic Jensen's gap c_{JG}.

A. Gamma distribution

Gamma distribution generalizes some of the commonly used fading models, including Rayleigh and Nakagami fading. The probability density function for Gamma distribution is given by

$$f\left(w\right)=\frac{w^{k-1}e^{-\frac{w}{\theta}}}{\theta^{k}\Gamma(k)} \tag{8}$$

for $w>0$, where $k>0$ is the shape parameter, and $\theta>0$ is the scale parameter.

Proposition 3. *If the elements of* Φ *are Gamma distributed with shape parameter* k*, they satisfy Equation* (4) *with constant* $c = \frac{\log(e)}{\alpha} - \log\left(1 + \frac{1}{2\alpha}\right)$ *for any* $0 < \alpha \leq k$*.*

Proof: Using Lemma 2, it is sufficient to prove $\log\left(\mathbb{E}\left[W\right]\right) - \mathbb{E}\left[\log\left(W\right)\right] \leq \frac{\log(e)}{\alpha} - \log\left(1 + \frac{1}{2\alpha}\right)$. It is known for the Gamma distribution that $\mathbb{E}\left[W\right] = k\theta$ and $\mathbb{E}\left[\ln\left(W\right)\right] = \psi\left(k\right) + \ln\left(\theta\right)$, where ψ is the digamma function. Therefore

$$\log\left(\mathbb{E}\left[W\right]\right) - \mathbb{E}\left[\log\left(W\right)\right] = \log(e)\left(\ln\left(k\right) - \psi\left(k\right)\right) \tag{9}$$

We first use the following property of digamma function

$$\psi\left(k\right) = \psi\left(k+1\right) - \frac{1}{k}, \tag{10}$$

and then use the inequality from [4]

$$\ln\left(k + \frac{1}{2}\right) < \psi\left(k+1\right) < \ln\left(k + e^{-\gamma}\right). \tag{11}$$

Hence

$$\begin{aligned} \log\left(\mathbb{E}\left[W\right]\right) - \mathbb{E}\left[\log\left(W\right)\right] &< \log(e)\left(\ln\left(k\right) - \ln\left(k + \frac{1}{2}\right) + \frac{1}{k}\right) \\ &= \frac{\log\left(e\right)}{\alpha} - \log\left(1 + \frac{1}{2\alpha}\right). \end{aligned} \tag{12}$$

The last step follows because the function involved is decreasing in k in the range $(0, \infty)$ and since it is assumed $0 < \alpha \leq k$. ■

Corollary 4. *If the elements of* Φ *are are exponentially distributed (which corresponds to Rayleigh fading), they satisfy Equation* (4) *with constant* $c = 0.86$*.*

Proof: In Rayleigh fading model the $\left|g_{ij}\right|^2$ is exponentially distributed. The exponential distribution itself is a special case of Gamma distribution with $k = 1$. Substituting $\alpha = 1$ in (12) we get $\log\left(\mathbb{E}\left[W\right]\right) - \mathbb{E}\left[\log\left(W\right)\right] > -0.86$. ■

Nakagami fading can be obtained as a special case of the Gamma distribution; in this case the logarithmic Jensen's gap will depend upon the parameters used in the model.

B. Weibull distribution

The probability density function for Weibull distribution is given by

$$f\left(w\right) = \frac{k}{\lambda}\left(\frac{w}{\lambda}\right)^{k-1} e^{-(w/\lambda)^k} \tag{13}$$

for $x > 0$ with $k, \lambda > 0$.

Proposition 5. *If the elements of* Φ *are Weibull distributed with parameter* k*, they satisfy Equation* (4) *with* $c = \frac{\gamma \log(e)}{\alpha} + \log\left(\Gamma\left(1 + \frac{1}{\alpha}\right)\right)$ *for any* $0 < \alpha \leq k$*, where* γ *is Euler's constant.*

Proof: For Weibull distributed W, we have $\mathbb{E}[W] = \lambda\Gamma\left(1 + \frac{1}{k}\right)$ and $\mathbb{E}[\ln(W)] = \ln(\lambda) - \frac{\gamma}{k}$, where $\Gamma(\cdot)$ denotes the gamma function and γ is the Euler's constant. Hence for $0 < \alpha \leq k$, it follows that

$$\log(\mathbb{E}[W]) - \mathbb{E}[\log(W)] \leq \frac{\gamma \log(e)}{\alpha} + \log\left(\Gamma\left(1 + \frac{1}{\alpha}\right)\right). \tag{14}$$

Using Lemma 2 concludes the proof. ■

Note that exponential distribution can be specialized from Weibull distribution as well, by setting $k = 1$. Hence we get the tighter gap in the following corollary.

Corollary 6. *If the elements of* Φ *are exponentially distributed, they satisfy Equation* (4) *with constant* $c = 0.83$.

In the following table we summarize the values we obtain as upper bound on logarithmic Jensen's gap, according to Definition 1 and Equation (4) for different distributions.

Table I

UPPER BOUND OF LOGARITHMIC JENSEN'S GAP FOR DIFFERENT DISTRIBUTIONS

Fading Model	c
Rayleigh	0.83
Gamma $k = 1$	0.86
Gamma $k = 2$	0.40
Gamma $k = 3$	0.26
Weibull $k = 1$	0.83
Weibull $k = 2$	0.24
Weibull $k = 3$	0.11

C. Other distributions

Here we give a lemma that can be used together with Lemma 2 to verify whether a given fading model has a finite logarithmic Jensen's gap.

Lemma 7. *If the cumulative distribution function* $F(w)$ *of* W *satisfies* $F(w) \leq aw^b$ *over* $w \in [0, \epsilon]$ *for some* $a \geq 0$, $b > 0$, *and* $0 < \epsilon \leq 1$, *then*

$$\mathbb{E}\left[\ln(W)\right] \geq \ln(\epsilon) + a\epsilon^b \ln(\epsilon) - \frac{a\epsilon^b}{b}. \tag{15}$$

Proof: The condition in this lemma ensures that the probability density function $f(w)$ grows slow enough as $w \to 0^-$ so that $f(w)\ln(w)$ is integrable at 0. Also the behavior for large values of w is not relevant here, since we are looking for a lower bound on $\mathbb{E}\left[\ln(W)\right]$. The detailed proof is in Appendix A. ■

Hence if the cumulative distribution of the channel gain grows polynomially in a neighborhood of 0, the resulting logarithm becomes integrable, and thus it is possible to find a finite constant c for the Equation (4).

In the following sections we make use of the logarithmic Jensen's gap characterization to derive approximate capacity results for FF-ICs.

IV. Approximate Capacity Region of FF-IC without feedback

In this section we make use of the logarithmic Jensen's gap characterization to obtain the approximate capacity region of FF-IC with neither feedback nor CSIT.

Theorem 8. *For a non-feedback FF-IC with a finite logarithmic Jensen's gap* c_{JG} *, the rate region* $\mathcal{R}_{NFB}$ *described by* $(16a) - (16g)$ *is achievable with* $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1\right)$*:*

$$R_1 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 1 \tag{16a}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 1 \tag{16b}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \tag{16c}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 2 \tag{16d}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + |g_{12}|^2\right)\right] - 2 \tag{16e}$$

$$2R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 3 \tag{16f}$$

$$R_1 + 2R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 3 \tag{16g}$$

and the region $\mathcal{R}_{NFB}$ has a capacity gap of at most $c_{JG}+1$ bits/s/Hz.

Proof: This region is obtained by a rate-splitting scheme that allocates the private message power proportional to $\frac{1}{\mathbb{E}[inr]}$. The analysis of the scheme and outer bounds are similar to that in [7]. See subsection IV-B for details. ■

Corollary 9. *Instantaneous CSIT cannot improve the capacity region of the FF-IC except by a constant.*

Proof: Our outer bounds in subsection IV-B for the non-feedback IC allow for instantaneous CSIT at the receivers. These outer bounds are within constant gap of the rate region $\mathcal{R}_{NFB}$ achieved without CSIT. Hence the proof. ■

Corollary 10. *Delayed CSIT cannot improve the capacity region of the FF-IC except by a constant.*

Proof: This follows from the previous corollary, since instantaneous CSIT is always better than delayed CSIT. ■

Remark 11. The previous two corollaries are for FF-IC with 2 users and single antennas. It does not contradict the results for MISO broadcast channel, X-channel, MIMO IC and multi-user IC where delayed CSIT or instantaneous CSIT can improve capacity region by more than a constant [12], [13], [11], [15], [26].

Corollary 12. *Within a constant gap, the capacity region of the FF-IC without feedback is same as the capacity region of IC (without fading) with equivalent channel strengths $SNR_i := \mathbb{E}\left[|g_{ii}|^2\right]$ for $i=1,2$, and $INR_i := \mathbb{E}\left[|g_{ij}|^2\right]$ for $i \neq j$.*

Proof: This is an application of the logarithmic Jensen's gap result. The rate region of non-feedback case in given in Equations (16a) to (16g) can be reduced to the rate region for a channel without fading. Let $\mathcal{R}'_{NFB}$ be the approximately optimal Han-Kobayashi rate region of IC [7] with equivalent channel strengths $SNR_i := \mathbb{E}\left[|g_{ii}|^2\right]$ for $i=1,2$, and $INR_i := \mathbb{E}\left[|g_{ij}|^2\right]$ for $i \neq j$. Then for a constant c' we have

$$\mathcal{R}'_{NFB} \supseteq \mathcal{R}_{NFB} \supseteq \mathcal{R}'_{NFB} - c'. \tag{17}$$

This can be verified by proceeding through each inner bound equation.For example, consider the first inner bound Equation (16a) $R_1 \leq \mathbb{E}\left[\log\left(1+|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]-1$. The corresponding equation in $\mathcal{R}'_{NFB}$ is $R_1 \leq \log\left(1+SNR_1+\lambda_{p2}INR_1\right)-1$. Now

$$\log\left(1+SNR_1+\lambda_{p2}INR_1\right)-1 \overset{(a)}{\leq} \mathbb{E}\left[\log\left(1+|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]-1 \tag{18}$$

$$\overset{(b)}{\leq} \left(\log\left(1+SNR_1+\lambda_{p2}INR_1\right)-1\right)-2c_{JG} \tag{19}$$

where (a) is due to Jensen's inequality and (b) is by applying logarithmic Jensen's gap result twice. Due to (18), (19) it follows that the first inner bound equation for fading case is in constant gap with that of static case. Similarly by proceeding through each inner bound equation, it follows that

$$\mathcal{R}'_{NFB} \subseteq \mathcal{R}_{NFB} \subseteq \mathcal{R}'_{NFB} - c'$$

for a constant c'. ∎

A. Discussion

It is useful to view Theorem 8 in the context of the existing results for the ICs. It is known that for ICs, one can approximately achieve the capacity region by performing superposition coding and allocating a power to the private symbols that is inversely proportional to the strength of the interference caused at the unintended receiver. Consequently, the received interference power is at the noise level, and the private symbols can be safely treated as noise, incurring only a constant rate penalty. At first sight, such a strategy seems impossible for the fading IC, where the transmitters do not have instantaneous channel information. What Theorem 8 reveals (with the details in subsection IV-B) is that if the fading model has finite logarithmic Jensen's gap, it is sufficient to perform power allocation based on the inverse of average interference strength to approximately achieve the capacity region.

We compare the symmetric rate point achievable for the non-feedback symmetric FF-IC in Figure 4. The fading model used is Rayleigh fading. The inner bound in numerical simulation is from Equations (20a) to (20g) in subsection IV-B according to the choice of distributions given in the same subsection. The outer bound is plotted by simulating Equations (24a) to (24g) in subsection IV-B. The SNR is varied after fixing $\frac{\log(INR)}{\log(SNR)}$. The simulation yields a capacity gap of 1.48 bits/s/Hz for $\alpha=0.5$ and a capacity gap of 1.51 bits/s/Hz for $\alpha=0.25$. Our theoretical analysis for FF-IC gives a capacity gap of $c_{JG}+1 \leq 1.83$ bits/s/Hz independent of α, using data from Table I in Section III.

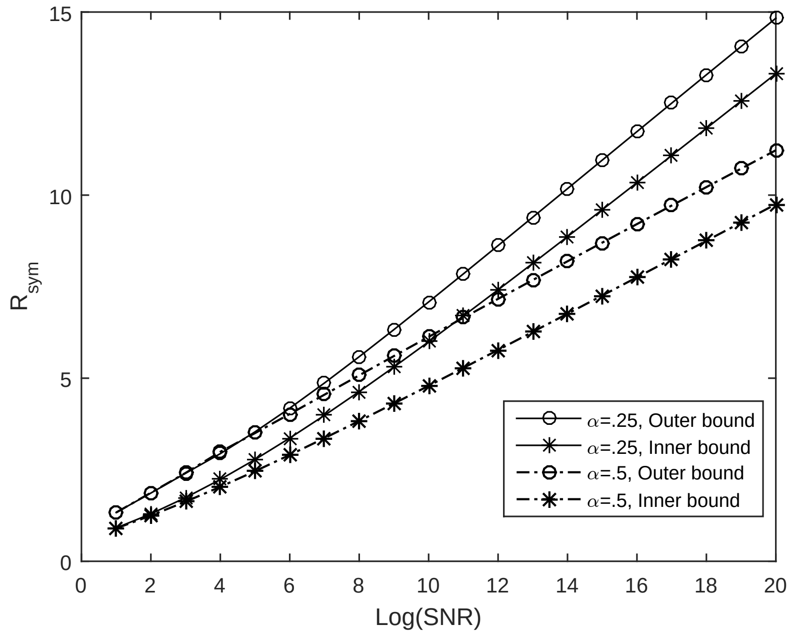

Figure 4. Comparison of outer and inner bounds with given $\alpha = \frac{\log(INR)}{\log(SNR)}$ for non-feedback symmetric FF-IC at the symmetric rate point. For high SNR, the capacity gap is approximately 1.48 bits/s/Hz for $\alpha = 0.5$ and 1.51 bits/s/Hz for $\alpha = 0.25$ from the numerics. Our theoretical analysis yields gap as 1.83 bits/s/Hz independent of α.

B. Proof of Theorem 8

From [6] we obtain that a Han-Kobayashi scheme for IC can achieve the following rate region for all $p\left(u_1\right)p\left(u_2\right)p\left(x_1|u_1\right)p\left(x_2|u_2\right)$. Note that we use $\left(Y_i, \underline{g_i}\right)$ instead of $\left(Y_i\right)$ in the actual result from [6] to account for the fading.

$$R_1 \leq I\left(X_1; Y_1, \underline{g_1}|U_2\right) \tag{20a}$$

$$R_2 \leq I\left(X_2; Y_2, \underline{g_2}|U_1\right) \tag{20b}$$

$$R_1 + R_2 \leq I\left(X_2, U_1; Y_2, \underline{g_2}\right) + I\left(X_1; Y_1, \underline{g_1}|U_1, U_2\right) \tag{20c}$$

$$R_1 + R_2 \leq I\left(X_1, U_2; Y_1, \underline{g_1}\right) + I\left(X_2; Y_2, \underline{g_2}|U_1, U_2\right) \tag{20d}$$

$$R_1 + R_2 \leq I\left(X_1, U_2; Y_1, \underline{g_1}|U_1\right) + I\left(X_2, U_1; Y_2, \underline{g_2}|U_2\right) \tag{20e}$$

$$2R_1 + R_2 \leq I\left(X_1, U_2; Y_1, \underline{g_1}\right) + I\left(X_1; Y_1, \underline{g_1}|U_1, U_2\right) + I\left(X_2, U_1; Y_2, \underline{g_2}|U_2\right) \tag{20f}$$

$$R_1 + 2R_2 \leq I\left(X_2, U_1; Y_2, \underline{g_2}\right) + I\left(X_2; Y_2, \underline{g_2}|U_1, U_2\right) + I\left(X_1, U_2; Y_1, \underline{g_1}|U_1\right). \tag{20g}$$

Now similar to that in [7], choose the Gaussian input distribution

$$U_k \sim \mathcal{CN}\left(0, \lambda_{ck}\right), \quad X_{pk} \sim \mathcal{CN}\left(0, \lambda_{pk}\right), \qquad k \in \{1, 2\} \tag{21}$$

$$X_1 = U_1 + X_{p1} \tag{22}$$

$$X_2 = U_2 + X_{p2} \tag{23}$$

where $\lambda_{ck} + \lambda_{pk} = 1$ and $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1\right)$. Here we introduced the rate-splitting using the average inr. On evaluating the region described by $(20a) - (20g)$ with this choice of input distribution, we get the region described by $(16a) - (16g)$; the calculations are deferred to Appendix B.

Claim 13. An outer bound for the non-feedback case is given by $(24a) - (24g)$

$$R_1 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2\right)\right] \tag{24a}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2\right)\right] \tag{24b}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{|g_{11}|^2}{1 + |g_{12}|^2}\right)\right] \tag{24c}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{|g_{22}|^2}{1 + |g_{21}|^2}\right)\right] \tag{24d}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{21}|^2 + \frac{|g_{11}|^2}{1 + |g_{12}|^2}\right)\right] + \mathbb{E}\left[\log\left(1 + |g_{12}|^2 + \frac{|g_{22}|^2}{1 + |g_{21}|^2}\right)\right] \tag{24e}$$

$$\begin{aligned} 2R_1 + R_2 \leq\ & \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + |g_{12}|^2 + \frac{|g_{22}|^2}{1 + |g_{21}|^2}\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \frac{|g_{11}|^2}{1 + |g_{12}|^2}\right)\right] \end{aligned} \tag{24f}$$

$$\begin{aligned} R_1 + 2R_2 \leq\ & \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + |g_{21}|^2 + \frac{|g_{11}|^2}{1 + |g_{12}|^2}\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \frac{|g_{22}|^2}{1 + |g_{21}|^2}\right)\right]. \end{aligned} \tag{24g}$$

Proof: The outer bounds (24a) and (24b) are easily derived by removing the interference from the other user by providing it as side-information.

The outer bound in Equation (24e) follows from [7, Theorem 1]. Those in Equation (24f) and Equation (24g) follow from [7, Theorem 4]. We just need to modify the theorems from [7] for the fading case by treating $\left(Y_i, \underline{g}_i\right)$ as output, and using the i.i.d property of the channels. We illustrate the procedure for Equation (24g) in Appendix C. Equation (24e) and Equation (24f) can be derived similarly.

The derivation of outer bounds (24c) and (24d) uses similar techniques as for Equation (24g). We derive Equation (24d) in Appendix C. Equation (24d) follows due to symmetry. ■

Claim 14. The gap between the inner bound $(16a) - (16g)$ and the outer bound $(24a) - (24g)$ for the feedback case is at most $c_{JG} + 1$ bits/s/Hz.

Proof: The proof for the capacity gap uses the logarithmic Jensen's gap property of the fading model. Denote the gap between the first outer bound $(24a)$ and first inner bound $(16a)$ by δ_1, δ_2 for the second pair and so on. By inspection $\delta_1 \leq 1$ and $\delta_2 \leq 1$. Now

$$\delta_3 = \mathbb{E}\left[\log\left(1 + \frac{|g_{11}|^2}{1 + |g_{12}|^2}\right)\right] - \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] + 2 \tag{25}$$

$$\stackrel{(a)}{\leq} \mathbb{E}\left[\log\left(1 + \frac{|g_{11}|^2}{1 + INR_1}\right)\right] - \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2\right)\right] + 2 + c_{JG} \tag{26}$$

$$\stackrel{(b)}{\leq} 2 + c_{JG} \tag{27}$$

The step (a) follows from Jensen's inequality and logarithmic Jensen's gap property of $|g_{12}|^2$. The step (b) follows because $\lambda_{p1} = \min\left(\frac{1}{INR_1}, 1\right) \geq \frac{1}{INR_1+1}$. Similarly we can bound the other δ's and gather the inequalities as:

$$\delta_1, \delta_2 \leq 1 \tag{28}$$

$$\delta_3, \delta_4 \leq 2 + c_{JG} \tag{29}$$

$$\delta_5 \leq 2 + 2c_{JG} \tag{30}$$

$$\delta_6, \delta_7 \leq 3 + 2c_{JG} \tag{31}$$

For δ_5, δ_6, and δ_7 we have to use the logarithmic Jensen's gap property twice and hence $2c_{JG}$ appears. We note that δ_1 is associated with bounding R_1, δ_2 with R_2, $(\delta_3, \delta_4, \delta_5)$ with $R_1 + R_2$, δ_6 with $2R_1 + R_2$ and δ_7 with $R_1 + 2R_2$. Hence it follows that the capacity gap is at most $\max\left(\delta_1, \delta_2, \frac{\delta_3}{2}, \frac{\delta_4}{2}, \frac{\delta_5}{2}, \frac{\delta_6}{3}, \frac{\delta_7}{3}\right) \leq c_{JG} + 1$ bits/s/Hz. ■

V. Approximate Capacity Region of FF-IC with feedback

In this section we make use of the logarithmic Jensen's gap characterization to obtain the approximate capacity region of FF-IC with output and channel state feedback, but transmitters having no prior knowledge of channel states. Under the feedback model, after each reception, each receiver reliably feeds back the received symbol and the channel states to its corresponding transmitter. For example, at time l, Tx1 receives $(Y_1(l-1), g_{11}(l-1), g_{21}(l-1))$ from Rx1. Thus $X_1(l)$ is allowed to be a function of $\left(W_1, \{Y_1(k), g_{11}(k), g_{21}(k)\}_{k<l}\right)$. The model is described in section II and is illustrated with Figure 2 in the same section.

Theorem 15. *For a feedback FF-IC with a finite logarithmic Jensen's gap c_{JG} , the rate region $\mathcal{R}_{FB}$ described by $(32a)-(32f)$ is achievable for $0 \le |\rho|^2 \le 1$ with $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1-|\rho|^2\right)$:*

$$R_1 \le \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \mathit{Re}\left(g_{11}g_{21}^*\right) + 1\right)\right] - 1 \tag{32a}$$

$$R_1 \le \mathbb{E}\left[\log\left(1 + \left(1-|\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \tag{32b}$$

$$R_2 \le \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2|\rho|^2 \mathit{Re}\left(g_{22}^*g_{12}\right) + 1\right)\right] - 1 \tag{32c}$$

$$R_2 \le \mathbb{E}\left[\log\left(1 + \left(1-|\rho|^2\right)|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 2 \tag{32d}$$

$$\begin{aligned} R_1 + R_2 \le\ & \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2|\rho|^2 \mathit{Re}\left(g_{22}^*g_{12}\right) + 1\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \end{aligned} \tag{32e}$$

$$\begin{aligned} R_1 + R_2 \le\ & \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \mathit{Re}\left(g_{11}g_{21}^*\right) + 1\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 2 \end{aligned} \tag{32f}$$

and the region $\mathcal{R}_{FB}$ has a capacity gap of at most $c_{JG} + 2$ bits/s/Hz.

Proof: The proof is in subsection V-A. ∎

Corollary 16. *Instantaneous CSIT cannot improve the capacity region of the FF-IC with feedback and delayed CSIT except for a constant.*

Proof: Our outer bounds in subsection V-A allow for feedback and instantaneous CSIT at the receivers. These outer bounds are within constant gap of the rate region $\mathcal{R}_{FB}$ achieved using only feedback and delayed CSIT. Hence the proof. ∎

Corollary 17. *Within a constant gap, the capacity region of the FF-IC with feedback and delayed CSIT is same as the capacity region of a feedback IC (without fading) with equivalent channel strengths $SNR_i := \mathbb{E}\left[|g_{ii}|^2\right]$ for $i = 1, 2$, and $INR_i := \mathbb{E}\left[|g_{ij}|^2\right]$ for $i \neq j$.*

Proof: This is again an application of the logarithmic Jensen's gap result. The proof is given in Appendix D. ∎

A. *Proof of Theorem 15*

Note that since the receivers know their respective incoming channel states, we can view the effective channel output at Rxi as the pair $\left(Y_i, \underline{g}_i\right)$. Then the block Markov scheme of [21, Lemma 1] implies that the rate pairs (R_1, R_2) satisfying

$$R_1 \leq I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \tag{33a}$$

$$R_1 \leq I\left(U_1; Y_2, \underline{g_2}|U, X_2\right) + I\left(X_1; Y_1, \underline{g_1}|U_1, U_2, U\right) \tag{33b}$$

$$R_2 \leq I\left(U, U_1, X_2; Y_2, \underline{g_2}\right) \tag{33c}$$

$$R_2 \leq I\left(U_2; Y_1, \underline{g_1}|U, X_1\right) + I\left(X_2; Y_2, \underline{g_2}|U_1, U_2, U\right) \tag{33d}$$

$$R_1 + R_2 \leq I\left(X_1; Y_1, \underline{g_1}|U_1, U_2, U\right) + I\left(U, U_1, X_2; Y_2, \underline{g_2}\right) \tag{33e}$$

$$R_1 + R_2 \leq I\left(X_2; Y_2, \underline{g_2}|U_1, U_2, U\right) + I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \tag{33f}$$

for all $p(u)\, p(u_1|u)\, p(u_2|u)\, p(x_1|u_1, u)\, p(x_2|u_2, u)$ are achievable. We choose the input distribution according to

$$U \sim \mathcal{CN}\left(0, |\rho|^2\right), U_k \sim \mathcal{CN}\left(0, \lambda_{ck}\right), X_{pk} \sim \mathcal{CN}\left(0, \lambda_{pk}\right) \tag{34}$$

$$X_1 = U + U_1 + X_{p1} \tag{35}$$

$$X_2 = U + U_2 + X_{p2} \tag{36}$$

with $0 \leq |\rho|^2 \leq 1$, $\lambda_{ck} + \lambda_{pk} = 1 - |\rho|^2$ and $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1 - |\rho|^2\right)$.

With this choice of λ_{pk} we perform the rate-splitting according to the average inr in place of rate-splitting based on the constant inr. On evaluating the terms in $(33a) - (33f)$ for this choice of input distribution, we get the inner bound described by $(32a) - (32f)$; the calculations are deferred to Appendix E.

An outer bound for the feedback case is given by $(37a) - (37f)$ with $0 \leq |\rho| \leq 1$:

$$R_1 \leq \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2\text{Re}\left(\rho g_{11} g_{21}^*\right) + 1\right)\right] \tag{37a}$$

$$R_1 \leq \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{11}|^2}{1 + \left(1 - |\rho|^2\right)|g_{12}|^2}\right)\right] \tag{37b}$$

$$R_2 \leq \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2\text{Re}\left(\rho g_{22}^* g_{12}\right) + 1\right)\right] \tag{37c}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{22}|^2}{1 + \left(1 - |\rho|^2\right)|g_{21}|^2}\right)\right] \tag{37d}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2\text{Re}\left(\rho g_{22}^* g_{12}\right) + 1\right)\right]$$

$$+ \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{11}|^2}{1 + \left(1 - |\rho|^2\right)|g_{12}|^2}\right)\right] \tag{37e}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2\text{Re}\left(\rho g_{11} g_{21}^*\right) + 1\right)\right]$$

$$+ \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{22}|^2}{1 + \left(1 - |\rho|^2\right)|g_{21}|^2}\right)\right], \tag{37f}$$

The outer bounds can be easily derived following the proof techniques from [21, Theorem 3] using $\mathbb{E}\left[X_1 X_2^*\right] = \rho$, treating $\left(Y_i, \underline{g}_i\right)$ as output, and using the i.i.d property of the channels. The calculations are deferred to deferred to Appendix F.

Claim 18. The gap between the inner bound $(32a) - (32f)$ and the outer bound $(37a) - (37f)$ for the feedback case is at most $c_{JG} + 2$ bits/s/Hz.

Proof: Denote the gap between the first outer bound and inner bound by δ_1, for the second pair denote the gap by δ_2, and so on. We have

$$\delta_1 = \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2\text{Re}\left(\rho g_{11} g_{21}^*\right) + 1\right)\right]$$

$$- \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \text{Re}\left(g_{11} g_{21}^*\right) + 1\right)\right] + 1 \tag{38}$$

$$\overset{(a)}{=} \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|g_{11}||g_{21}||\rho|\cos(\theta) + 1\right)\right]$$

$$- \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|g_{11}||g_{21}||\rho|^2\cos(\theta) + 1\right)\right] + 1 \tag{39}$$

$$\overset{(b)}{=} \mathbb{E}\left[\log\left(\frac{\left(1 + |g_{11}|^2 + |g_{21}|^2\right)^2 + \sqrt{\left(1 + |g_{11}|^2 + |g_{21}|^2\right)^2 - \left(2|\rho||g_{11}||g_{21}|\right)^2}}{\left(1 + |g_{11}|^2 + |g_{21}|^2\right)^2 + \sqrt{\left(1 + |g_{11}|^2 + |g_{21}|^2\right)^2 - \left(2|\rho|^2|g_{11}||g_{21}|\right)^2}}\right)\right] + 1 \tag{40}$$

$$\overset{(c)}{\le} 1 \tag{41}$$

where (a) is because phases of g_{11}, g_{12} are independently uniformly distributed in $[0, 2\pi]$ yielding $\text{Re}\left(g_{11} g_{21}^*\right) = |g_{11}||g_{21}|\cos(\theta)$ with an independent $\theta \sim$ Unif $[0, 2\pi]$, (b) is using the fact that for $p > q$ $\frac{1}{2\pi}\int_0^{2\pi} \log\left(p + q\cos(\theta)\right) d\theta = \log\left(\frac{p + \sqrt{p^2 - q^2}}{2}\right)$ and (c) is because the numerator in the fraction is less than the denominator.

Now we consider the gap δ_2 between the second inequality $(37b)$ of the outer bound and the second inequality $(32b)$ of the inner bound.

$$\delta_2 = \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{11}|^2}{1 + \left(1 - |\rho|^2\right)|g_{12}|^2}\right)\right] - \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] + 2 \tag{42}$$

$$\overset{(a)}{\leq} \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)INR_1+\left(1-|\rho|^2\right)|g_{11}|^2\right)\right]-\log\left(1+\left(1-|\rho|^2\right)INR_1\right)+c_{JG}$$
$$-\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]+2 \tag{43}$$
$$\leq \mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)INR_1}\right)\right]-\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2\right)\right]+2+c_{JG} \tag{44}$$
$$\overset{(b)}{\leq} 2+c_{JG} \tag{45}$$

where (a) follows by using logarithmic Jensen's gap property on $|g_{12}|^2$ and Jensen's inequality. The step (b) follows because

$$\frac{\left(1-|\rho|^2\right)}{1+\left(1-|\rho|^2\right)INR_1}=\frac{1}{\frac{1}{1-|\rho|^2}+INR_1}\leq \min\left(\frac{1}{INR_1},1-|\rho|^2\right)=\lambda_{p1}. \tag{46}$$

Similarly by inspection of the other bounding inequalities we can gather the inequalities on the δ's as:

$$\delta_1,\delta_3\leq 1 \tag{47}$$
$$\delta_2,\delta_4,\delta_5,\delta_6\leq c_{JG}+2 \tag{48}$$

We note that (δ_1,δ_2) is associated with bounding R_1, (δ_3,δ_4) with R_2, (δ_5,δ_6) with R_1+R_2. Hence it follows that the capacity gap is at most $\max\left(\delta_1,\delta_2,\delta_3,\delta_4,\frac{\delta_5}{2},\frac{\delta_6}{2}\right)\leq c_{JG}+2$ bits/s/Hz.

■

VI. Approximate capacity of feedback FF-IC using point-to-point codes

As the third illustration for the usefulness of logarithmic Jensen's gap, we propose a strategy that does not make use of rate-splitting, superposition coding or joint decoding for the feedback case, which achieves the entire capacity region for 2-user symmetric FF-ICs to within a constant gap. This constant gap is dictated by the logarithmic Jensen's gap for the fading model. Our scheme only uses point-to-point codes, and a feedback scheme based on amplify-and-forward relaying, similar to the one proposed in [21].

The main idea behind the scheme is to have one of the transmitters initially send a very densely modulated block of data, and then refine this information using feedback and amplify-and-forward relaying for the following blocks, in a fashion similar to the Schalkwijk-Kailath scheme [18], while treating the interference as noise. Such refinement effectively induces a 2-tap point-to-point inter-symbol-interference (ISI) channel at the unintended receiver, and a point-to-point feedback channel for the intended receiver. As a result, both receivers can decode their intended information using only point-to-point codes.

Consider the symmetric fading interference channel, where the channel statistics are symmetric and independent, *i.e.*, $g_{ii}(l) \sim g_d$ and $g_{ij}(l) \sim g_c$, for $i \neq j$. We consider n transmission phases, each phase having a block length of N. For Tx1, generate 2^{nNR_1} codewords $\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$ i.i.d according to $\mathcal{CN}(0,1)$. Tx1 encodes its message $W_1 \in \left\{1, \ldots, 2^{nNR_1}\right\}$ onto $\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$. For Tx2, generate 2^{nNR_2} codewords $X_2^{(1)N} = X_2^N$ i.i.d according to $\mathcal{CN}(0,1)$ and let it encode its message $W_2 \in \left\{1, \ldots, 2^{nNR_2}\right\}$ onto $X_2^{(1)N} = X_2^N$. Note that for Tx2 the coding block length is N, whereas it is nN for Tx1.

Tx1 sends $X_1^{(i)N}$ in phase i. Tx2 sends $X_2^{(1)N} = X_2^N$ in phase 1. At the beginning of phase $i > 1$, Tx2 receives

$$Y_2^{(i-1)N} = g_{22}^{(i-1)N} X_2^{(i-1)N} + g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N} \tag{49}$$

from feedback. It can remove $g_{22}^{(i-1)N} X_2^{(i-1)N}$ from $Y_2^{(i-1)N}$ to obtain $g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}$. Tx2 then transmits the resulting interference-plus-noise after power scaling as $X_2^{(i)N}$, i.e.

$$X_2^{(i)N} = \frac{g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}}. \tag{50}$$

Thus in phase $i > 1$, Rx2 receives

$$Y_2^{(i)N} = g_{22}^{(i)N} X_2^{(i)N} + g_{12}^{(i)N} X_1^{(i)N} + Z_2^{(i)N} \tag{51}$$

$$= g_{22}^{(i)N} \left(\frac{g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}} \right) + g_{12}^{(i)N} X_1^{(i)N} + Z_2^{(i)N} \tag{52}$$

and feeds it back to Tx2 for phase $i+1$. The transmission scheme is summarized in Table II. Note that for phase $i = 1$ Tx1 receives

$$Y_1^{(1)N} = g_{11}^{(1)N} X_1^{(1)N} + g_{21}^{(1)N} X_2^{(1)N} + Z_1^{(1)N} \tag{53}$$

and for phase $i > 1$ Tx1 observes a block ISI channel since it receives

$$Y_1^{(i)N} = g_{11}^{(i)N} X_1^{(i)N} + g_{21}^{(i)N} \left(\frac{g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}} \right) + Z_1^{(i)N} \tag{54}$$

$$= g_{11}^{(i)N} X_1^{(i)N} + \left(\frac{g_{21}^{(i)N} g_{12}^{(i-1)N}}{\sqrt{1+INR}} \right) X_1^{(i-1)N} + \tilde{Z}_1^{(i)N} \tag{55}$$

where $\tilde{Z}_1^{(i)N} = Z_1^{(i)N} + \frac{g_{21}^{(i)N} Z_2^{(i-1)N}}{\sqrt{1+INR}}$.

Table II

TRANSMITTED SYMBOLS IN n-PHASE SCHEME FOR SYMMETRIC FF-IC WITH FEEDBACK

User	Phase 1	Phase 2	.	.	Phase n
1	$X_1^{(1)N}$	$X_1^{(2)N}$	.	.	$X_1^{(n)N}$
2	$X_2^{(1)N}$	$\frac{g_{12}^{(1)N}X_1^{(1)N}+Z_2^{(1)N}}{\sqrt{1+INR}}$	.	.	$\frac{g_{12}^{(n-1)N}X_1^{(n-1)N}+Z_2^{(n-1)N}}{\sqrt{1+INR}}$

At the end of n blocks, Rx1 collects $\mathbf{Y_1}^N = \left(Y_1^{(1)N}, \ldots, Y_1^{(n)N}\right)$ and decodes W_1 such that $\left(\mathbf{X_1}^N(W_1), \mathbf{Y_1}^N\right)$ is jointly typical (where $\mathbf{X_1}^N = \left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$) treating $X_2^{(1)N} = X_2^N$ as noise. At Rx2, channel outputs over n phases can be combined with an appropriate scaling so that the interference-plus-noise at phases $\{1, \ldots, n-1\}$ are successively canceled, *i.e.*, an effective point-to-point channel can be generated through $\tilde{Y}_2^N = \sum_{i=1}^n \left(\prod_{j=i+1}^n \frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right) Y_2^{(i)N}$ (see the analysis in the subsection VI-A for details). Note that this can be viewed as a block version of the Schalkwijk-Kailath scheme [18] (and the references therein). Given the effective channel $\tilde{Y}_2^N$, the receiver can simply use point-to-point typicality decoding to recover W_2, treating the interference in phase n as noise.

Theorem 19. *For a symmetric FF-IC with a finite logarithmic Jensen's gap c_{JG} , the rate pair*

$$(R_1, R_2) = \left(\log\left(1 + SNR + INR\right) - 3c_{JG} - 2, \mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right]\right)$$

is achievable by the scheme. The scheme together with switching the roles of users and time-sharing, achieves the capacity region of symmetric feedback IC within $3c_{JG} + 2$ bits/s/Hz.

Proof: The proof follows from the analysis in the following subsection. ■

A. Analysis of Point-to-Point Codes for Symmetric FF-ICs

We now provide the analysis for the scheme, going through the decoding at the two receivers and then looking at the capacity gap for the achievable region.

1) Decoding at Rx1 : At the end of n blocks Rx1 collects $\mathbf{Y_1}^N = \left(Y_1^{(1)N}, \ldots Y_1^{(n)N}\right)$ and decodes W_1 such that $\left(\mathbf{X_1}^N(W_1), \mathbf{Y_1}^N\right)$ is jointly typical, where $\mathbf{X_1}^N = \left(X_1^{(1)N}, \ldots X_1^{(n)N}\right)$. The joint typicality is considered according the product distribution $p^N(\mathbf{X_1}, \mathbf{Y_1})$, where

$$p(\mathbf{X_1}, \mathbf{Y_1}) = p\left(\left(X_1^{(1)}, \ldots X_1^{(n)}\right), \left(Y_1^{(1)}, \ldots Y_1^{(n)}\right)\right) \tag{56}$$

is a joint Gaussian distribution, dictated by the following equations that arise from our n-phase scheme:

$$Y_1^{(1)} = g_{11}^{(1)} X_1^{(1)} + g_{21}^{(1)} X_2^{(1)} + Z_1^{(1)} \tag{57}$$

And for $i = 2, 3, \ldots, n$:

$$Y_1^{(i)} = g_{11}^{(i)} X_1^{(i)} + g_{21}^{(i)} \left(\frac{g_{12}^{(i-1)} X_1^{(i-1)} + Z_2^{(i-1)}}{\sqrt{1+INR}} \right) + Z_1^{(i)} \tag{58}$$

with $X_1^{(i)}, X_2^{(1)}, Z_1^{(i)}$ being i.i.d $\mathcal{CN}(0,1)$. Essentially $X_2^{(1)}, Z_1^{(i)}$ are both Gaussian noise for Rx1.

Using standard techniques it follows that for the n-phase scheme as $N \to \infty$ user 1 can achieve the rate $\frac{1}{n}\mathbb{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)\right]$ where $|K_{\mathbf{Y_1}}(n)|$ denotes the determinant of covariance matrix for the the n-phase scheme defined in the following pattern

$$K_{\mathbf{Y_1}}(1) = \left[1 + |g_{11}(1)|^2 + |g_{21}(1)|^2\right]$$

$$K_{\mathbf{Y_1}}(2) = \begin{bmatrix} |g_{11}(2)|^2 + \frac{|g_{21}(2)|^2\left(|g_{12}(1)|^2+1\right)}{1+INR} + 1 & \frac{g_{11}^*(1) g_{21}(2) g_{12}(1)}{\sqrt{1+INR}} \\ \frac{g_{11}(1) g_{21}^*(2) g_{12}^*(1)}{\sqrt{1+INR}} & |g_{11}(1)|^2 + |g_{21}(1)|^2 + 1 \end{bmatrix}$$

$$K_{\mathbf{Y_1}}(3) = \begin{bmatrix} |g_{11}(3)|^2 + \frac{|g_{21}(3)|^2\left(|g_{12}(2)|^2+1\right)}{1+INR} + 1 & \frac{g_{11}^*(2) g_{21}(3) g_{12}(2)}{\sqrt{1+INR}} & 0 \\ \frac{g_{11}(2) g_{21}^*(3) g_{12}^*(2)}{\sqrt{1+INR}} & |g_{11}(2)|^2 + \frac{|g_{21}(2)|^2\left(|g_{12}(1)|^2+1\right)}{1+INR} + 1 & \frac{g_{11}^*(1) g_{21}(2) g_{12}(1)}{\sqrt{1+INR}} \\ 0 & \frac{g_{11}(1) g_{21}^*(2) g_{12}^*(1)}{\sqrt{1+INR}} & |g_{11}(1)|^2 + |g_{21}(1)|^2 + 1 \end{bmatrix}$$

where $g_{11}(i) \sim g_d$ i.i.d and $g_{12}(i), g_{21}(i) \sim g_c$ i.i.d. Letting $n \to \infty$, Rx1 can achieve the rate $R_1 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)\right]$. We need to evaluate $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)\right]$. The following lemma gives an lower bound on $\frac{1}{n}\mathbb{E}\left[\log\left(|K_{\mathbf{Y_1}}(n)|\right)\right]$.

Lemma 20.

$$\frac{1}{n}\mathbb{E}\left[\log\left(|K_{\mathbf{Y_1}}(n)|\right)\right] \geq \frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) - 3c_{JG}$$

where $\hat{K}_{\mathbf{Y_1}}(n)$ is obtained from $K_{\mathbf{Y_1}}(n)$ by replacing $g_{12}(i)$'s,$g_{21}(i)$'s with $\sqrt{INR}$ and $g_{11}(i)$'s with $\sqrt{SNR}$.

Proof: The proof involves expanding the matrix determinant and repeated application of the logarithmic Jensen's gap property. The details are given in Appendix G. ∎

Subsequently we use the following lemma in bounding $\lim_{n\to\infty} \frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right)$.

Lemma 21. *If* $A_1 = [|a|]$, $A_2 = \begin{bmatrix} |a| & b \\ b^* & |a| \end{bmatrix}$, $A_3 = \begin{bmatrix} |a| & b & 0 \\ b^* & |a| & b \\ 0 & b^* & |a| \end{bmatrix}$, $A_4 = \begin{bmatrix} |a| & b & 0 & 0 \\ b^* & |a| & b & 0 \\ 0 & b^* & |a| & b \\ 0 & 0 & b^* & |a| \end{bmatrix}$ *etc. with* $|a|^2 > 4|b|^2$*, then*

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(|A_n|\right) \geq \log\left(\frac{|a|}{2}\right).$$

Proof: The proof is given in Appendix H. ■

For the n-phase scheme, the $\left|\hat{K}_{\mathbf{Y_1}}(n)\right|$ matrix has the form A_n, as defined in Lemma 21 after identifying $|a| = 1 + INR + SNR$ and $b = \frac{\sqrt{SNR}INR}{\sqrt{1+INR}}$. Note that with this choice $|a|^2 > 4|b|^2$ holds due to AM-GM (Arithmetic Mean $\geq$ Geometric Mean) inequality. Hence we have

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) \geq \log\left(\frac{1 + INR + SNR}{2}\right) \tag{59}$$

using Lemma 21. Also, $K_{\mathbf{Y_1}|\mathbf{X_1}}(n)$ is a diagonal matrix of the form

$$K_{\mathbf{Y_1}|\mathbf{X_1}}(n) = \text{diag}\left(\frac{|g_{21}(n)|^2}{1+INR} + 1, \frac{|g_{21}(n-1)|^2}{1+INR} + 1, \ldots, \frac{|g_{21}(2)|^2}{1+INR} + 1, |g_{21}(1)|^2 + 1\right) \tag{60}$$

Hence using Jensen's inequality

$$\limsup_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\log\left(\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|\right)\right] \leq \limsup_{n\to\infty} \frac{1}{n} \log\left(\left(\frac{INR}{1+INR} + 1\right)^{n-1} (1 + INR)\right) \tag{61}$$

$$= \log\left(\frac{INR}{1+INR} + 1\right) \tag{62}$$

$$\leq 1 \tag{63}$$

Hence by combining Lemma 20, Equation (59) and Equation (63), we get

$$R_1 \leq \log\left(1 + INR + SNR\right) - 3c_{JG} - 2 \tag{64}$$

is achievable.

2) Decoding at Rx2 *:* For user 2 we can use a block variant of Schalkwijk-Kailath scheme [18] to achieve $R_2 = \mathbb{E}\left[\log^+\left(\frac{|g_d|^2}{1+INR}\right)\right]$. The key idea is that the interference-plus-noise sent in subsequent slots can indeed refine the symbols of the previous slot. The chain of refinement over n phases compensate for the fact that the information symbols are sent only in the first phase. We have

$$Y_2^{(1)N} = g_{22}^{(1)N} X_2^N + g_{12}^{(1)N} X_1^{(1)N} + Z_2^{(1)N} \tag{65}$$

and

$$Y_2^{(i)N} = g_{22}^{(i)N}\left(\frac{g_{12}^{(i-1)N}X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}}\right) + g_{12}^{(i)N}X_1^{(i)N} + Z_2^{(i)N} \tag{66}$$

for $i > 1$. Now let $\tilde{Y}_2^N = \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n}\frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right)Y_2^{(i)N}$. We have

$$\begin{aligned}
\tilde{Y}_2^N &= \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n}\frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right)Y_2^{(i)N} && (67)\\
&= g_{22}^{(n)N}\left(\frac{g_{12}^{(n-1)N}X_1^{(n-1)N} + Z_2^{(n-1)N}}{\sqrt{1+INR}}\right) + g_{12}^{(n)N}X_1^{(n)N} + Z_2^{(n)N}\\
&\quad + \left(\frac{-g_{22}^{(n)N}}{\sqrt{1+INR}}\right)\left(g_{22}^{(n-1)N}\left(\frac{g_{12}^{(n-2)N}X_1^{(n-2)N} + Z_2^{(n-2)N}}{\sqrt{1+INR}}\right) + g_{12}^{(n-1)N}X_1^{(n-1)N} + Z_2^{(n-1)N}\right)\\
&\quad + \left(\frac{g_{22}^{(n)N}g_{22}^{(n-1)N}}{1+INR}\right)\left(g_{22}^{(n-2)N}\left(\frac{g_{12}^{(n-3)N}X_1^{(n-3)N} + Z_2^{(n-3)N}}{\sqrt{1+INR}}\right) + g_{12}^{(n-2)N}X_1^{(n-2)N} + Z_2^{(n-2)N}\right)\\
&\quad + \cdots\\
&\quad + \left(\prod_{j=2}^{n}\frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right)\left(g_{22}^{(1)N}X_2^N + g_{21}^{(1)N}X_2^{(1)N} + Z_1^{(1)N}\right) && (68)\\
&= g_{22}^{(1)N}\left(\prod_{j=2}^{n}\frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right)X_2^N + g_{12}^{(n)N}X_1^{(n)N} + Z_2^{(n)N}. && (69)
\end{aligned}$$

due to cross-cancellation. Now Rx2 decodes for its message from $\tilde{Y}_2^N$. Hence Rx2 can achieve the rate

$$R_2 \leq \liminf_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(1 + \left(\prod_{j=2}^{n}\frac{|g_{22}(j)|^2}{1+INR}\right)\frac{|g_{22}(1)|^2}{1+|g_{12}(n)|^2}\right)\right] \tag{70}$$

where $g_{22}(1),\ldots,g_{22}(n) \sim g_d$ being i.i.d and $g_{12}(n) \sim g_c$. Hence it follows that

$$R_2 \leq \mathbb{E}\left[\log^+\left(\frac{|g_d|^2}{1+INR}\right)\right] \tag{71}$$

is achievable.

3) Capacity gap: We can obtain the following outer bounds from Theorem 15 for the special case of symmetric fading statistics.

$$R_1, R_2 \leq \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 1\right)\right] \tag{72}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + \frac{|g_d|^2}{1+|g_c|^2}\right)\right] + \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 2|g_d||g_c| + 1\right)\right] \tag{73}$$

where Equation (72) is obtained from Equation (37b) and Equation (37d) by setting $\rho = 0$ (note that $\rho = 0$ yields the loosest version of outer bounds in Equation (37b) and Equation (37d)). Similarly Equation (73) is a looser version of outer bound Equation (37e) independent of ρ. The outer bounds reduce to a pentagonal region with two non-trivial corner points (see Figure 5). Our n-phase scheme can achieve the two corner points within $2 + 3c_{JG}$ bits/s/Hz for each user. The proof is using logarithmic Jensen's gap property and is deferred to Appendix I.

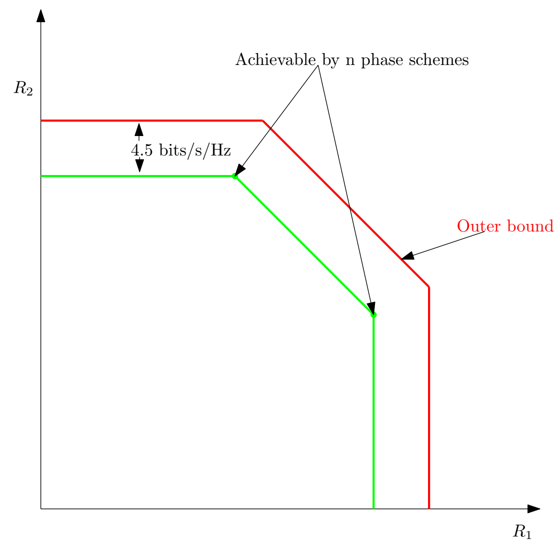

Figure 5. Illustration of bounds for capacity region for symmetric FF-IC. The corner points of the outer bound can be approximately achieved by our n-phase schemes. The gap is approximately 4.5 bits/s/Hz for the Rayleigh fading case.

B. *An auxiliary result: Approximate capacity of 2-tap fast Fading ISI channel*

Consider the 2-tap fast fading ISI channel described by

$$Y(l) = g_d(l)\, X(l) + g_c(l)\, X(l-1) + Z(l)\,, \tag{74}$$

where $g_d \sim \mathcal{CN}(0, SNR)$ and $g_d \sim \mathcal{CN}(0, INR)$ are independent fading known only to the receiver and $Z \sim \mathcal{CN}(0,1)$. Also we assume a power constraint of $\mathbb{E}\left[|X|^2\right] \leq 1$ on the transmit symbols. Our analysis for R_1 can be easily modified to obtain a closed form approximate expression for this channel. This gives rise to the following corollary on the capacity of fading ISI channels.

Corollary 22. *The capacity C_{F-ISI} of the 2-tap fast fading ISI channel is bounded by*

$$\log(1 + SNR + INR) + 1 \geq C_{F-ISI} \geq \log(1 + SNR + INR) - 1 - 3c_{JG},$$

where the channel fading strengths is assumed to have a logarithmic Jensen's gap of c_{JG}.

Proof: The proof is given in Appendix J. ■

VII. Conclusion

We introduced the notion of logarithmic Jensen's gap and demonstrated that it can be used to obtain approximate capacity region for FF-ICs. We proved that the rate-splitting schemes for ICs [7], [6], [21], when extended to the fast fading case give capacity gap as a function of the logarithmic Jensen's gap. Our analysis of logarithmic Jensen's gap for fading models like Rayleigh fading show that rate-splitting is approximately optimal for such cases. We then developed a scheme for symmetric FF-ICs, which can be implemented using point-to-point codes and can approximately achieve the capacity region. An important direction to study will be to see if similar schemes with point-to-point codes can be extended to general FF-ICs. Also our schemes are not approximately optimal for bursty IC since it does not have finite logarithmic Jensen's gap, it would be interesting to study if the schemes can be extended to bursty IC and then to any arbitrary fading distribution.

Appendix A
Proof of Lemma 7

We have $F(w) \leq aw^b$ for $w \in [0,\epsilon]$ where $a \geq 0, b > 0, 1 \geq \epsilon > 0$. Now using integration by parts we get

$$\mathbb{E}[\ln(W)] \geq \int_0^1 f(w)\ln(w) \tag{75}$$

$$= \int_0^\epsilon f(w)\ln(w) + \int_\epsilon^1 f(w)\ln(w) \tag{76}$$

$$= [F(w)\ln(w)]_0^\epsilon - \int_0^\epsilon F(w)\frac{1}{w} + \int_\epsilon^1 f(w)\ln(w) \tag{77}$$

$$\geq \left[aw^b\ln(w)\right]_0^\epsilon - \int_0^\epsilon aw^b\frac{1}{w} + \ln(\epsilon) \tag{78}$$

$$\geq a\epsilon^b\ln(\epsilon) - \frac{a\epsilon^b}{b} + \ln(\epsilon). \tag{79}$$

Note that $\ln(w)$ is negative in the range $[0,1)$, thus we get the desired inequalities in the last two steps.

Appendix B
Proof of achievability for non-feedback case

We evaluate the term in the first inner bound inequality $(20a)$. The other terms can be similarly evaluated.

$$I\left(X_1;Y_1,\underline{g_1}|U_2\right) \overset{(a)}{=} I\left(X_1;Y_1|\,U_2,\underline{g_1}\right) \tag{80}$$

$$= h\left(Y_1|\,U_2,\underline{g_1}\right) - h\left(Y_1|\,X_1,U_2,\underline{g_1}\right) \tag{81}$$

$$h\left(Y_1|\,U_2,\underline{g_1}\right) = h\left(g_{11}X_1 + g_{21}X_2 + Z_1|\,U_2,\underline{g_1}\right) \tag{82}$$

$$= h\left(g_{11}X_1 + g_{21}X_{p2} + Z_1|\,\underline{g_1}\right) \tag{83}$$

$$\text{variance}\left(g_{11}X_1 + g_{21}X_{p2} + Z_1|\,\underline{g_1}\right) = |g_{11}|^2 + \lambda_{p2}\,|g_{21}|^2 + 1 \tag{84}$$

$$\therefore I\left(X_1;Y_1,\underline{g_1}|U_2\right) = \mathbb{E}\left[\log\left(|g_{11}|^2 + \lambda_{p2}\,|g_{21}|^2 + 1\right)\right] + \log\left(2\pi e\right) \tag{85}$$

$$h\left(Y_1|X_1,U_2,\underline{g_1}\right) = h\left(g_{11}X_1 + g_{21}X_2 + Z_1|\,X_1,U_2,\underline{g_1}\right) \tag{86}$$

$$= h\left(g_{21}X_{p2} + Z_1|\,\underline{g_1}\right) \tag{87}$$

$$= \mathbb{E}\left[\log\left(1 + \lambda_{p2}\,|g_{21}|^2\right)\right] + \log\left(2\pi e\right) \tag{88}$$

$$\overset{(b)}{\leq} \mathbb{E}\left[\log\left(1 + \frac{1}{INR_2}\,|g_{21}|^2\right)\right] + \log\left(2\pi e\right) \tag{89}$$

$$\overset{(c)}{\leq} \log\left(2\right) + \log\left(2\pi e\right) \tag{90}$$

$$= 1 + \log\left(2\pi e\right), \tag{91}$$

$$\therefore I\left(U,U_2,X_1;Y_1,\underline{g_1}\right) \geq \mathbb{E}\left[\log\left(|g_{11}|^2 + \lambda_{p2}\,|g_{21}|^2 + 1\right)\right] - 1 \tag{92}$$

where (a) uses independence, (b) is because $\lambda_{pi} \leq \frac{1}{INR_i}$, and (c) follows from Jensen's inequality.

Appendix C
Proof of outer bounds for non-feedback case

Note that we have the notation $\underline{g} = [g_{11}, g_{21}, g_{22}, g_{12}]$, $S_1 = g_{12}X_1 + Z_2$, and $S_2 = g_{21}X_2 + Z_1$. Our outer bounding steps are valid while allowing X_{1i} to be a function of $\left(W_1, \underline{g^n}\right)$, thus letting transmitters have instantaneous and future CSIT. On choosing a uniform distribution of messages we get

$$n(R_1 + 2R_2 - \epsilon_n)$$

$$\leq I\left(W_1;Y_1^n,\underline{g^n}\right) + I\left(W_2;Y_2^n,\underline{g^n}\right) + I\left(W_2;Y_2^n,\underline{g^n}\right) \tag{93}$$

$$= I\left(W_1;Y_1^n|\,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,\underline{g^n}\right) \tag{94}$$

$$\leq I\left(W_1;Y_1^n,S_1^n|\,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,\underline{g^n}\right)+I\left(W_2;Y_2^n,S_2^n|\,X_1^n,\underline{g^n}\right) \tag{95}$$

$$\begin{aligned}&= I\left(W_1;S_1^n|\,\underline{g^n}\right)+I\left(W_1;Y_1^n|\,S_1^n,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,\underline{g^n}\right)\\&\quad+I\left(W_2;S_2^n|\,X_1^n,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,X_1^n,S_2^n,\underline{g^n}\right)\end{aligned} \tag{96}$$

$$\begin{aligned}&= h\left(S_1^n|\,\underline{g^n}\right)-h\left(S_1^n|\,W_1,\underline{g^n}\right)+h\left(Y_1^n|\,S_1^n,\underline{g^n}\right)-h\left(Y_1^n|\,W_1,S_1^n,\underline{g^n}\right)\\&\quad+h\left(Y_2^n|\,\underline{g^n}\right)-h\left(Y_2^n|\,W_2,\underline{g^n}\right)+h\left(S_2^n|\,X_1^n,\underline{g^n}\right)-h\left(S_2^n|\,X_1^n,W_2,\underline{g^n}\right)\\&\quad+h\left(Y_2^n|\,X_1^n,S_2^n,\underline{g^n}\right)-h\left(Y_2^n|\,X_1^n,W_2,S_2^n,\underline{g^n}\right)\end{aligned} \tag{97}$$

$$\begin{aligned}&= h\left(S_1^n|\,\underline{g^n}\right)-h\left(Z_2^n\right)+h\left(Y_1^n|\,S_1^n,\underline{g^n}\right)-h\left(S_2^n|\,\underline{g^n}\right)+h\left(Y_2^n|\,\underline{g^n}\right)-h\left(S_1^n|\,\underline{g^n}\right)\\&\quad+h\left(S_2^n|\,\underline{g^n}\right)-h\left(Z_1^n\right)+h\left(Y_2^n|\,X_1^n,S_2^n,\underline{g^n}\right)-h\left(Z_2^n\right)\end{aligned} \tag{98}$$

$$= h\left(Y_1^n|\,S_1^n,\underline{g^n}\right)+h\left(Y_2^n|\,\underline{g^n}\right)+h\left(Y_2^n|\,X_1^n,S_2^n,\underline{g^n}\right)-h\left(Z_1^n\right)-2h\left(Z_2^n\right) \tag{99}$$

$$\begin{aligned}&\overset{(a)}{\leq}\sum\left[h\left(Y_{1i}|S_{1i},\underline{g^n}\right)-h\left(Z_{1i}\right)\right]+\sum\left[h\left(Y_{2i}|\underline{g^n}\right)-h\left(Z_{2i}\right)\right]\\&\quad+\sum\left[h\left(Y_{2i}|X_{1i},S_{2i},\underline{g^n}\right)-h\left(Z_{1i}\right)\right]\end{aligned} \tag{100}$$

$$\begin{aligned}&= \mathbb{E}_{\underline{g^n}}\left[\sum\left(h\left(Y_{1i}|S_{1i},\underline{g^n}\right)-h\left(Z_{1i}\right)\right)\right]+\mathbb{E}_{\underline{g^n}}\left[\sum\left(h\left(Y_{2i}|\underline{g^n}\right)-h\left(Z_{2i}\right)\right)\right]\\&\quad+\mathbb{E}_{\underline{g^n}}\left[\sum\left(h\left(Y_{2i}|X_{1i},S_{2i},\underline{g^n}\right)-h\left(Z_{1i}\right)\right)\right]\end{aligned} \tag{101}$$

$$\begin{aligned}&\overset{(b)}{\leq} n\mathbb{E}\left[\log\left(1+|g_{21}|^2+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right]+n\mathbb{E}\left[\log\left(1+|g_{12}|^2+|g_{22}|^2\right)\right]\\&\quad+n\mathbb{E}\left[\log\left(1+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right]\end{aligned} \tag{102}$$

where (a) is due to the fact that conditioning reduces entropy and (b) follows from Equations [7, (50)] , [7, (51)] and [7, (52)]. Note that in the calculation of step (b) we allow the symbols X_{1i}, X_{2i} to depend on $\underline{g^n}$, but since $\underline{g^n}$ is available in conditioning the calculation proceeds similar to that in [7].

$$n(R_1+R_2-\epsilon_n) \tag{103}$$

$$\leq I\left(W_2;Y_2^n,\underline{g^n}\right)+I\left(W_2;Y_2^n,\underline{g^n}\right) \tag{104}$$

$$= I\left(W_1;Y_1^n|\,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,\underline{g^n}\right) \tag{105}$$

$$\leq I\left(W_1;Y_1^n|\,\underline{g^n}\right)+I\left(W_2;Y_2^n,S_2^n|\,X_1^n,\underline{g^n}\right) \tag{106}$$

$$= I\left(W_1;Y_1^n|\,\underline{g^n}\right)+I\left(W_2;S_2^n|\,X_1^n,\underline{g^n}\right)+I\left(W_2;Y_2^n|\,X_1^n,S_2^n,\underline{g^n}\right) \tag{107}$$

$$= h\left(Y_1^n \middle| \underline{g}^n\right) - h\left(Y_1^n \middle| W_1, \underline{g}^n\right) + h\left(S_2^n \middle| X_1^n, \underline{g}^n\right) - h\left(S_2^n \middle| X_1^n, W_2, \underline{g}^n\right)$$

$$+ h\left(Y_2^n \middle| X_1^n, S_2^n, \underline{g}^n\right) - h\left(Y_2^n \middle| X_1^n, W_2, S_2^n, \underline{g}^n\right) \tag{108}$$

$$= h\left(Y_1^n \middle| \underline{g}^n\right) - h\left(S_2^n \middle| \underline{g}^n\right) + h\left(S_2^n \middle| \underline{g}^n\right) - h\left(Z_1^n\right) + h\left(Y_2^n \middle| X_1^n, S_2^n, \underline{g}^n\right) - h\left(Z_2^n\right) \tag{109}$$

$$= h\left(Y_1^n \middle| \underline{g}^n\right) + h\left(Y_2^n \middle| X_1^n, S_2^n, \underline{g}^n\right) - h\left(Z_1^n\right) - h\left(Z_2^n\right) \tag{110}$$

$$\overset{(a)}{\leq} \sum \left[h\left(Y_{1i}|\underline{g}^n\right) - h\left(Z_{1i}\right)\right] + \sum \left[h\left(Y_{2i}|X_{1i}, S_{2i}, \underline{g}^n\right) - h\left(Z_{1i}\right)\right] \tag{111}$$

$$= \mathbb{E}_{\underline{g}^n}\left[\sum \left(h\left(Y_{1i}|\underline{g}^n\right) - h\left(Z_{1i}\right)\right)\right] + \mathbb{E}_{\underline{g}^n}\left[\sum \left(h\left(Y_{2i}|X_{1i}, S_{2i}, \underline{g}^n\right) - h\left(Z_{1i}\right)\right)\right] \tag{112}$$

$$\overset{(b)}{\leq} n\mathbb{E}\left[\log\left(1 + |g_{21}|^2 + |g_{11}|^2\right)\right] + n\mathbb{E}\left[\log\left(1 + \frac{|g_{22}|^2}{1 + |g_{21}|^2}\right)\right] \tag{113}$$

where (a) is due to the fact that conditioning reduces entropy and (b) again follows from Equations [7, (51)] and [7, (52)].

Appendix D
Proof of Corollary 17

Let $\mathcal{R}'_{NFB}$ be the approximately optimal Han-Kobayashi rate region of feedback IC [21] with equivalent channel strengths $SNR_i := \mathbb{E}\left[|g_{ii}|^2\right]$ for $i = 1, 2$, and $INR_i := \mathbb{E}\left[|g_{ij}|^2\right]$ for $i \neq j$. Then for a constant c'' we have

$$\mathcal{R}'_{FB} \supseteq \mathcal{R}_{FB} \supseteq \mathcal{R}'_{FB} - c''. \tag{114}$$

This can be verified by proceeding through each inner bound equation. For example, consider the first inner bound Equation 32a $R_1 \leq \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \text{Re}\left(g_{11}g_{21}^*\right) + 1\right)\right] - 1$. The corresponding equation in $\mathcal{R}'_{NFB}$ is $R_1 \leq \log\left(1 + SNR_1 + INR_2 + 2|\rho|^2 \sqrt{SNR_1 \cdot INR_2} + 1\right) - 1$. Now

$$\mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \text{Re}\left(g_{11}g_{21}^*\right) + 1\right)\right]$$

$$\overset{(a)}{\leq} \log\left(1 + SNR_1 + INR_2\right) \tag{115}$$

$$\leq \log\left(1 + SNR_1 + INR_2 + 2|\rho|^2 \sqrt{SNR_1 \cdot INR_2} + 1\right) \tag{116}$$

where (a) is due to Jensen's inequality and independence of g_{11}, g_{21}. Also

$$\mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \text{Re}\left(g_{11}g_{21}^*\right) + 1\right)\right]$$

$$\overset{(a)}{=} \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|g_{11}||g_{21}||\rho|\cos(\theta) + 1\right)\right] \tag{117}$$

$$\overset{(b)}{\geq} \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 1\right)\right] - 1 \tag{118}$$

$$\overset{(c)}{\geq} \log\left(SNR_1 + INR_2 + 1\right) - 1 - 2c_{JG} \tag{119}$$

$$\overset{(d)}{\geq} \log\left(SNR_1 + INR_2 + 2\left|\rho\right|^2 \sqrt{SNR_1 \cdot INR_2} + 1\right) - 2 - 2c_{JG} \tag{120}$$

where (a) is because phases of g_{11}, g_{12} are independently uniformly distributed in $[0, 2\pi]$ yielding Re $(g_{11}g_{21}^*) = |g_{11}|\,|g_{21}|\cos(\theta)$ with an independent $\theta \sim$ Unif $[0, 2\pi]$, (b) is using the fact that for $p > q$ $\frac{1}{2\pi}\int_0^{2\pi} \log\left(p + q\cos(\theta)\right) d\theta = \log\left(\frac{p+\sqrt{p^2-q^2}}{2}\right) \geq \log(p) - 1$, (c) is using the logarithmic Jensen's gap result twice and (d) is because $SNR_1 + INR_2 \geq 2\left|\rho\right|^2 \sqrt{SNR_1 \cdot INR_2}$. It follows from Equations (116) and (120), that the first inner bound for fading case is within constant gap with the first inner bound of the static case.

Now consider the second inner bound Equation 32b

$$R_1 \leq \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \tag{121}$$

and the corresponding equation

$$R_1 \leq \log\left(1 + \left(1 - |\rho|^2\right) INR_1\right) + \log\left(1 + \lambda_{p1}SNR_1 + \lambda_{p2}INR_2\right) - 3c_{JG} - 2 \tag{122}$$

from $\mathcal{R}'_{FB}$. We have

$$\begin{aligned}&\mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right]\\ &\leq \log\left(1 + \left(1 - |\rho|^2\right) INR_1\right) + \log\left(1 + \lambda_{p1}SNR_1 + \lambda_{p2}INR_2\right)\end{aligned} \tag{123}$$

due to Jensen's inequality. And

$$\begin{aligned}&\mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right]\\ &\geq \log\left(1 + \left(1 - |\rho|^2\right) INR_1\right) + \log\left(1 + \lambda_{p1}SNR_1 + \lambda_{p2}INR_2\right) - 3c_{JG}\end{aligned} \tag{124}$$

using the logarithmic Jensen's gap result thrice. It follows from Equations (123) and (124), that the second inner bound for fading case is within constant gap with the second inner bound of the static case. Similarly by proceeding through each inner bound equation, it follows that

$$\mathcal{R}'_{FB} \subseteq \mathcal{R}_{FB} \subseteq \mathcal{R}'_{FB} - c''$$

for a constant c''.

APPENDIX E
PROOF OF ACHIEVABILITY FOR FEEDBACK CASE

We evaluate the term in the first inner bound inequality $(33a)$. The other terms can be similarly evaluated.

$$I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \overset{(a)}{=} I\left(U, U_2, X_1; Y_1 | \, \underline{g_1}\right) \tag{125}$$

$$= h\left(Y_1 | \underline{g_1}\right) - h\left(Y_1 | \underline{g_1}, U, U_2, X_1\right), \tag{126}$$

$$\text{variance}\left(Y_1 | \underline{g_1}\right) = \text{variance}\left(g_{11}X_1 + g_{21}X_2 + Z_1 | \, g_{11}, g_{21}\right) \tag{127}$$

$$= |g_{11}|^2 + |g_{21}|^2 + g_{11}^* g_{21} \mathbb{E}\left[X_1^* X_2\right] + g_{11} g_{21}^* \mathbb{E}\left[X_1 X_2^*\right] + 1 \tag{128}$$

$$= |g_{11}|^2 + |g_{21}|^2 + 2\,|\rho|^2 \,\text{Re}\left(g_{11} g_{21}^*\right) + 1 \tag{129}$$

$$h\left(Y_1 | \underline{g_1}, U, U_2, X_1\right) = h\left(g_{11}X_1 + g_{21}X_2 + Z_1 | \, \underline{g_1}, U, U_2, X_1\right) \tag{130}$$

$$= h\left(g_{21}X_{p2} + Z_1 | \, \underline{g_1}\right) \tag{131}$$

$$= \mathbb{E}\left[\log\left(1 + \lambda_{p2}\,|g_{21}|^2\right)\right] + \log\left(2\pi e\right) \tag{132}$$

$$\overset{(b)}{\leq} \mathbb{E}\left[\log\left(1 + \frac{1}{INR_2}\,|g_{21}|^2\right)\right] + \log\left(2\pi e\right) \tag{133}$$

$$\overset{(c)}{\leq} \log\left(2\right) + \log\left(2\pi e\right) \tag{134}$$

$$= 1 + \log\left(2\pi e\right), \tag{135}$$

$$\therefore I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \geq \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2\,|\rho|^2 \,\text{Re}\left(g_{11} g_{21}^*\right) + 1\right)\right] - 1 \tag{136}$$

where (a) uses independence, (b) is because $\lambda_{pi} \leq \frac{1}{INR_i}$, and (c) follows from Jensen's inequality.

APPENDIX F
PROOF OF OUTER BOUNDS FOR FEEDBACK CASE

Following the methods in [21], we let $\mathbb{E}\left[X_1 X_2^*\right] = \rho$. We have the notation $\underline{g_1} = [g_{11}, g_{21}]$, $\underline{g_2} = [g_{22}, g_{12}]$, $\underline{g} = [g_{11}, g_{21}, g_{22}, g_{12}]$, $S_1 = g_{12}X_1 + Z_2$, and $S_2 = g_{21}X_2 + Z_1$. We let $\mathbb{E}\left[X_1 X_2^*\right] = \rho = |\rho|\, e^{i\theta}$. All of our outer bounding steps are valid while allowing X_{1i} to be a function of $\left(W_1, Y_1^{i-1}, \underline{g_1^n}\right)$, thus letting transmitters have full CSIT along with feedback. On choosing a uniform distribution of messages we get

$$n(R_1 - \epsilon_n) \overset{(a)}{\leq} I\left(W_1; Y_1^n | \, \underline{g_1^n}\right) \tag{137}$$

$$\overset{(b)}{\leq} \sum \left(h\left(Y_{1i}|\underline{g_{1i}}\right) - h\left(Z_{1i}\right)\right) \tag{138}$$

$$= \sum \left(\mathbb{E}_{\underline{\tilde{g}_{1i}}}\left[h\left(Y_{1i}|\underline{g_{1i}} = \underline{\tilde{g}_{1i}}\right) - h\left(Z_{1i}\right)\right]\right) \tag{139}$$

$$\overset{(c)}{=} \mathbb{E}_{\underline{\tilde{g}_1}}\left[\sum \left(h\left(Y_{1i}|\underline{g_{1i}} = \underline{\tilde{g}_1}\right) - h\left(Z_{1i}\right)\right)\right] \tag{140}$$

$$\therefore R_1 \leq \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + (\rho^* g_{11}^* g_{21} + \rho g_{11} g_{21}^*) + 1\right)\right] \tag{141}$$

where (a) follows from Fano's inequality, (b) follows from the fact that conditioning reduces entropy, and (c) follows from the fact that $\underline{\tilde{g}_{1i}}$ are i.i.d. Now we bound R_1 in a second way as done in [21]:

$$n(R_1 - \epsilon_n) \leq I\left(W_1; Y_1^n, \underline{g_1^n}\right) \tag{142}$$

$$\leq I\left(W_1; Y_1^n, \underline{g_1^n}, Y_2^n, \underline{g_2^n}, W_2\right) \tag{143}$$

$$= I\left(W_1; \underline{g}^n, W_2\right) + I\left(W_1; Y_1^n, Y_2^n | \underline{g}^n, W_2\right) \tag{144}$$

$$= 0 + I\left(W_1; Y_1^n, Y_2^n | \underline{g}^n, W_2\right) \tag{145}$$

$$= h\left(Y_1^n, Y_2^n | \underline{g}^n, W_2\right) - h\left(Y_1^n, Y_2^n | \underline{g}^n, W_1, W_2\right) \tag{146}$$

$$= \sum \left[h\left(Y_{1i}, Y_{2i} | \underline{g}^n, W_2, Y_1^{i-1}, Y_2^{i-1}\right)\right] - \sum \left[h\left(Z_{1i}\right) + h\left(Z_{2i}\right)\right] \tag{147}$$

$$= \sum \left[h\left(Y_{2i} | \underline{g}^n, W_2, Y_1^{i-1}, Y_2^{i-1}\right)\right] + \sum \left[h\left(Y_{1i} | \underline{g}^n, W_2, Y_1^{i-1}, Y_2^{i}\right)\right] - \sum \left[h\left(Z_{1i}\right) + h\left(Z_{2i}\right)\right] \tag{148}$$

$$\overset{(a)}{=} \sum \left[h\left(Y_{2i} | \underline{g}^n, W_2, Y_1^{i-1}, Y_2^{i-1}, X_2^i\right)\right] + \sum \left[h\left(Y_{1i} | \underline{g}^n, W_2, Y_1^{i-1}, Y_2^{i}, S_{1i}, X_2^i\right)\right] - \sum \left[h\left(Z_{1i}\right) + h\left(Z_{2i}\right)\right] \tag{149}$$

$$\overset{(b)}{\leq} \sum \left[h\left(Y_{2i} | \underline{g_i}, X_{2i}\right) - h\left(Z_{2i}\right)\right] + \sum \left[h\left(Y_{1i} | \underline{g_i}, S_{1i}, X_{2i}\right) - h\left(Z_{1i}\right)\right] \tag{150}$$

$$\overset{(c)}{=} \mathbb{E}_{\underline{\tilde{g}}}\left[\sum \left(h\left(Y_{2i} | X_{2i}, \underline{g_i} = \underline{\tilde{g}}\right) - h\left(Z_{2i}\right)\right)\right] + \mathbb{E}_{\underline{\tilde{g}}}\left[\sum \left(h\left(Y_{1i} | S_{1i}, X_{2i}, \underline{g_i} = \underline{\tilde{g}}\right) - h\left(Z_{1i}\right)\right)\right] \tag{151}$$

$$\therefore\ R_1 \overset{(d)}{\leq} \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{\left(1 - |\rho|^2\right)|g_{11}|^2}{1 + \left(1 - |\rho|^2\right)|g_{12}|^2}\right)\right] \tag{152}$$

where (a) follows from the fact that X_2^i is a function of $\left(W_2, Y_2^{i-1}, \underline{g}^n\right)$ and S_{1i} is a function of $\left(Y_2^i, X_2^i, \underline{g}^n\right)$, (b) follows from the fact that conditioning reduces entropy, (c) follows from the fact that $\underline{\tilde{g}_i}$ are i.i.d., and (d) follows from [21, (43)]. The other outer bounds can be derived similarly following [21] and making suitable changes to account for fading as we illustrated in the previous two derivations.

Appendix G
Fading matrix

The calculations are given in Equations (153),(154).

$$\begin{aligned}
&\mathbb{E}\left[\log\left(\left|K_{\mathbf{Y_1}}(n)\right|\right)\right] \\
&= \mathbb{E}\left[\log\left(\left(\left|g_{11}(n)\right|^2 + \left|g_{21}(n)\right|^2\left(\frac{\left|g_{12}(n-1)\right|^2+1}{1+INR}\right)+1\right)\left|K_{\mathbf{Y_1}}(n-1)\right|\right.\right. \\
&\qquad\left.\left. -\frac{\left|g_{11}(n-1)\right|^2\left|g_{21}(n)\right|^2\left|g_{12}(n-1)\right|^2}{1+INR}\left|K_{\mathbf{Y_1}}(n-2)\right|\right)\right] \qquad (153) \\
&\geq \mathbb{E}\left[\log\left((1+INR+SNR)\left|K_{\mathbf{Y_1}}(n-1)\right|\right.\right. \\
&\qquad\left.\left. -\frac{INR\cdot INR\left|g_{11}(n-1)\right|^2}{1+INR}\left|K_{\mathbf{Y_1}}(n-2)\right|\right)\right] - 3c_{JG} \qquad (154)
\end{aligned}$$

The first step (153), is by expanding the determinant. We use the logarithmic Jensen's gap property thrice in the second step (154). This is justified because the coefficients of $\left\{\left|g_{11}(n)\right|^2, \left|g_{12}(n-1)\right|^2, \left|g_{21}(n)\right|^2\right\}$ from Equation (153) are non-negative (due to the fact that all the matrices involved are covariance matrices), and the coefficients themselves are independent of $\left\{\left|g_{11}(n)\right|^2, \left|g_{12}(n-1)\right|^2, \left|g_{21}(n)\right|^2\right\}$. (Note that $\left|K_{\mathbf{Y_1}}(n-1)\right|$ depend on $\left|g_{12}(n-2)\right|^2$ but not on $\left|g_{12}(n-1)\right|^2$). This procedure can be carried out n times and it follows that:

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(\left|K_{\mathbf{Y_1}}(n)\right|\right)\right] \geq \lim_{n\to\infty}\frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) - 3c_{JG} \qquad (155)$$

where $\hat{K}_{\mathbf{Y_1}}(n)$ is obtained from $K_{\mathbf{Y_1}}(n)$ by replacing $g_{12}(i)$'s,$g_{21}(i)$'s with $\sqrt{INR}$ and $g_{11}(i)$'s with $\sqrt{SNR}$.

Appendix H
Matrix determinant: asymptotic behavior

The following recursion easily follows:

$$|A_n| = |a|\,|A_{n-1}| - |b|^2\,|A_{n-2}| \qquad (156)$$

with $|A_1| = |a|, |A_2| = |a|^2 - |b|^2$. Also $|A_0|$ can be consistently defined to be 1. The characteristic equation for this recursive relation is given by: $\lambda^2 - |a|\lambda + |b|^2 = 0$ and the characteristic roots are given by:

$$\lambda_1 = \frac{|a| + \sqrt{|a|^2 - 4|b|^2}}{2}, \lambda_2 = \frac{|a| - \sqrt{|a|^2 - 4|b|^2}}{2}. \tag{157}$$

Now the solution of the recursive system is given by:

$$|A_n| = c_1\lambda_1^n + c_2\lambda_2^n \tag{158}$$

with the boundary conditions

$$1 = c_1 + c_2 \tag{159}$$

$$|a| = c_1\lambda_1 + c_2\lambda_2. \tag{160}$$

It can be easily seen that $c_1 > 0$, $\lambda_1 > \lambda_2 > 0$ since $|a|^2 > 4|b|^2$ by assumption of Lemma 21. Now

$$\lim_{n\to\infty} \frac{1}{n} \log(|A_n|) = \lim_{n\to\infty} \frac{1}{n} \log(c_1\lambda_1^n + c_2\lambda_2^n) \tag{161}$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\log(\lambda_1^n) + \log\left(c_1 + c_2 \frac{\lambda_2^n}{\lambda_1^n} \right) \right) \tag{162}$$

$$\overset{(a)}{=} \log(\lambda_1) \tag{163}$$

$$= \log\left(\frac{|a| + \sqrt{|a|^2 - 4|b|^2}}{2} \right). \tag{164}$$

The step (a) follows because $1 > \frac{\lambda_2}{\lambda_1} > 0$ and $c_1 > 0$.

APPENDIX I

APPROXIMATE CAPACITY USING N PHASE SCHEMES

We have the following outer bounds from Theorem 15.

$$R_1, R_2 \leq \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 1\right)\right] \tag{165}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + \frac{|g_d|^2}{1 + |g_c|^2}\right)\right] + \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 2|g_d||g_c| + 1\right)\right] \tag{166}$$

The above outer bound region is a polytope with the following two non-trivial corner points:

$$\left\{ \begin{array}{lcc} R_1 & = & \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 1\right)\right] \\ R_2 & = & \mathbb{E}\left[\log\left(1 + \frac{|g_d|^2}{1+|g_c|^2}\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\right] \end{array} \right\}$$

$$\left\{ \begin{array}{lcl} R_1 & = & \mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right]+\mathbb{E}\left[\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\right] \\ R_2 & = & \mathbb{E}\left[\log\left(|g_d|^2+|g_c|^2+1\right)\right] \end{array} \right\}$$

We can achieve these rate points within $2+3c_{JG}$ bits/s/Hz for each user using the n-phase schemes since

$$(R_1,R_2)=\left(\log\left(1+SNR+INR\right)-2-3c_{JG},\mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right]\right) \tag{167}$$

$$(R_1,R_2)=\left(\mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right],\log\left(1+SNR+INR\right)-2-3c_{JG}\right) \tag{168}$$

are achievable and since using Jensen's inequality

$$\mathbb{E}\left[\log\left(|g_d|^2+|g_c|^2+1\right)\right]\leq\log\left(1+SNR+INR\right). \tag{169}$$

The only important point left to verify is in the following claim.

Claim 23. $\mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right]+\mathbb{E}\left[\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\right]-\mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right]\leq 2+c_{JG}$

Proof: We have $\frac{2|g_d||g_c|}{|g_d|^2+|g_c|^2}\leq 1$ due to AM-GM inequality. Hence

$$\mathbb{E}\left[\log\left(1+\frac{2\left|g_d\right|\left|g_c\right|}{1+\left|g_d\right|^2+\left|g_c\right|^2}\right)\right]\leq 1. \tag{170}$$

Also

$$\mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right]\leq\mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+INR}\right)\right]+c_{JG} \tag{171}$$

using logarithmic Jensen's gap property. Hence it only remains to show $\log\left(1+\frac{|g_d|^2}{1+INR}\right)-\log^+\left[\frac{|g_d|^2}{1+INR}\right]\leq 1$ to complete the proof.

If $\log^+\left[\frac{|g_d|^2}{1+INR}\right]=0$ then $\frac{|g_d|^2}{1+INR}\leq 1$ and hence $\log\left(1+\frac{|g_d|^2}{1+INR}\right)\leq\log\left(2\right)=1$.

If $\log^+\left[\frac{|g_d|^2}{1+INR}\right]>0$ then $\frac{|g_d|^2}{1+INR}>1$ and hence again

$$\log\left(1+\frac{|g_d|^2}{1+INR}\right)-\log^+\left[\frac{|g_d|^2}{1+INR}\right]=\log\left(1+\frac{1+INR}{|g_d|^2}\right)<1. \tag{172}$$

■

APPENDIX J
ANALYSIS FOR THE 2-TAP FADING ISI CHANNEL

We have for the outer bound

$$\begin{aligned}
n\left(R-\epsilon_n\right) &\le I\left(Y^n, g_d^n, g_c^n; W\right) && (173)\\
&= I\left(Y^n; W \middle| g_d^n, g_c^n\right) && (174)\\
&= h\left(Y^n \middle| g_d^n, g_c^n\right) - h\left(Z^n\right) && (175)\\
&\le \sum h\left(Y_i \middle| g_{d,i}, g_{c,i}\right) - h\left(Z^n\right) && (176)\\
&\overset{(a)}{\le} \sum \mathbb{E}\left[\log\left(1 + P_i \left|g_d\right|^2 + P_{i-1}\left|g_c\right|^2 + 2\left|g_d\right|\left|g_c\right|\sqrt{P_i P_{i-1}}\right)\right] && (177)\\
&\le \sum \left(\mathbb{E}\left[\log\left(1 + P_i\left|g_d\right|^2 + P_{i-1}\left|g_c\right|^2\right)\right] + 1\right) && (178)
\end{aligned}$$

where (a) is using P_i as the power for i^{th} symbol and using Cauchy Schwarz inequality to bound $\left|\mathbb{E}\left[X_i X_{i-1}\right]\right| \le \sqrt{P_i P_{i-1}}$. Now using Jensen's inequality it follows that

$$\begin{aligned}
R-\epsilon_n &\le \mathbb{E}\left[\log\left(1+\left|g_d\right|^2+\left|g_c\right|^2\right)\right]+1 && (179)\\
&\le \log\left(1+SNR+INR\right)+1. && (180)
\end{aligned}$$

For the inner bound similar to the scheme in subsection (VI), using Gaussian codebooks and n phases we obtain that

$$R = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(\frac{\left|K_{\mathbf{Y}}(n)\right|}{\left|K_{\mathbf{Y}|\mathbf{X}}(n)\right|}\right)\right] \tag{181}$$

is achievable where $\boldsymbol{X}(n)$ is n-length Gaussian vector with i.i.d $\mathcal{CN}\left(0,1\right)$ elements and $\boldsymbol{Y}\left(n\right)$ is generated from $\boldsymbol{X}(n)$ by the ISI channel (from Equation (74)). Here $\left|K_{\mathbf{Y}|\mathbf{X}}(n)\right| = \left|K_{\mathbf{Z}}(n)\right| = 1$ because Z is AWGN . Hence

$$R = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(\left|K_{\mathbf{Y}}(n)\right|\right)\right] \tag{182}$$

is achievable. Hence it follows that

$$R \ge \log\left(1+SNR+INR\right) - 1 - 3c_{JG} \tag{183}$$

is achievable due to Lemma 20 and Lemma 21.

REFERENCES

[1] V. Aggarwal, L. Sankar, A. R. Calderbank, and H. V. Poor. Ergodic layered erasure one-sided interference channels. In *IEEE Information Theory Workshop*, pages 574–578, Oct 2009.

[2] A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse. Wireless network information flow: A deterministic approach. *IEEE Transactions on Information Theory*, 57(4):1872–1905, April 2011.

[3] G. Bassi, P. Piantanida, and S. Yang. Capacity bounds for a class of interference relay channels. *IEEE Transactions on Information Theory*, 61(7):3698–3721, July 2015.

[4] N. Batir. Inequalities for the gamma function. *Archiv der Mathematik*, 91(6):554–563, 2008.

[5] V. R. Cadambe and S. A. Jafar. Interference alignment and degrees of freedom of the k-user interference channel. *IEEE Transactions on Information Theory*, 54(8):3425–3441, Aug 2008.

[6] H-F. Chong, M. Motani, H. K. Garg, and H. El Gamal. On the Han-Kobayashi region for the interference channel. *IEEE Transactions on Information Theory*, 54(7):3188–3194, 2008.

[7] R. H. Etkin, D. N. C. Tse, and H. Wang. Gaussian interference channel capacity to within one bit. *IEEE Transactions on Information Theory*, 54(12):5534–5562, 2008.

[8] R. K. Farsani. The capacity region of the wireless ergodic fading interference channel with partial CSIT to within one bit. In *IEEE International Symposium on Information Theory*, pages 759–763, July 2013.

[9] S. Gherekhloo, A. Chaaban, and A. Sezgin. Cooperation for interference management: A gdof perspective. *IEEE Transactions on Information Theory*, 62(12):6986–7029, Dec 2016.

[10] T. S. Han and K. Kobayashi. A new achievable rate region for the interference channel. *IEEE Transactions on Information Theory*, 27(1):49–60, Jan 1981.

[11] M. G. Kang and W. Choi. Ergodic interference alignment with delayed feedback. *IEEE Signal Processing Letters*, 20(5):511–514, May 2013.

[12] M. A. Maddah-Ali and D. N. C. Tse. On the degrees of freedom of miso broadcast channels with delayed feedback. *EECS Department, University of California, Berkeley, Tech. Rep. UCB/EECS-2010-122*, 2010.

[13] H. Maleki, S. A. Jafar, and S. Shamai. Retrospective interference alignment over interference networks. *IEEE Journal of Selected Topics in Signal Processing*, 6(3):228–240, June 2012.

[14] I. Maric, R. Dabora, and A. J. Goldsmith. Relaying in the presence of interference: Achievable rates, interference forwarding, and outer bounds. *IEEE Transactions on Information Theory*, 58(7):4342–4354, July 2012.

[15] B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath. Ergodic interference alignment. *IEEE Transactions on Information Theory*, 58(10):6355–6371, 2012.

[16] V. M. Prabhakaran and P. Viswanath. Interference channels with source cooperation. *IEEE Transactions on Information Theory*, 57(1):156–186, Jan 2011.

[17] O. Sahin and E. Erkip. Achievable rates for the gaussian interference relay channel. In *IEEE Global Telecommunications Conference (GLOBECOM)*, pages 1627–1631, Nov 2007.

[18] J. Schalkwijk and T. Kailath. A coding scheme for additive noise channels with feedback–i: No bandwidth constraint. *IEEE Transactions on Information Theory*, 12(2):172–182, 1966.

[19] J. Sebastian, C. Karakus, and S. N. Diggavi. Approximately achieving the feedback interference channel capacity with point-to-point codes. In *IEEE International Symposium on Information Theory*, pages 715–719, July 2016.

[20] J. Sebastian, C. Karakus, S. N. Diggavi, and I. H. Wang. Rate splitting is approximately optimal for fading gaussian interference channels. In *Annual Allerton Conference on Communication, Control, and Computing*, pages 315–321, Sept 2015.

[21] C. Suh and D. N. C. Tse. Feedback capacity of the gaussian interference channel to within 2 bits. *IEEE Transactions on Information Theory*, 57(5):2667–2685, May 2011.

[22] R. Tandon, S. Mohajer, H. V. Poor, and S. Shamai. Degrees of freedom region of the mimo interference channel with output feedback and delayed csit. *IEEE Transactions on Information Theory*, 59(3):1444–1457, March 2013.

[23] Y. Tian and A. Yener. The gaussian interference relay channel: Improved achievable rates and sum rate upperbounds using a potent relay. *IEEE Transactions on Information Theory*, 57(5):2865–2879, May 2011.

[24] A. Vahid, M. A. Maddah-Ali, and A. S. Avestimehr. Capacity results for binary fading interference channels with delayed csit. *IEEE Transactions on Information Theory*, 60(10):6093–6130, Oct 2014.

[25] A. Vahid, M. A. Maddah-Ali, A. S. Avestimehr, and Y. Zhu. Binary fading interference channel with no csit. *IEEE Transactions on Information Theory*, 63(6):3565–3578, June 2017.

[26] C. S. Vaze and M. K. Varanasi. The degrees of freedom region and interference alignment for the mimo interference channel with delayed csit. *IEEE Transactions on Information Theory*, 58(7):4396–4417, July 2012.

[27] H. Wang, C. Suh, S. N. Diggavi, and P. Viswanath. Bursty interference channel with feedback. In *IEEE International Symposium on Information Theory*, pages 21–25. IEEE, 2013.

[28] I. H. Wang and D. N. C. Tse. Interference mitigation through limited receiver cooperation. *IEEE Transactions on Information Theory*, 57(5):2913–2940, May 2011.

[29] I. H. Wang and D. N. C. Tse. Interference mitigation through limited transmitter cooperation. *IEEE Transactions on Information Theory*, 57(5):2941–2965, May 2011.

[30] Y. Zhu and D. Guo. Ergodic fading z-interference channels without state information at transmitters. *IEEE Transactions on Information Theory*, 57(5):2627–2647, May 2011.